

Answer Key

1. Chloride Nernst potential during development

Given: $[\text{Cl}^-]_o = 140 \text{ mM}$ and $\frac{RT}{F} = 25 \text{ mV}$. We use the Nernst equation for chloride in the form (consistent with the sign for an anion):

$$E_{\text{Cl}} = -\frac{RT}{F} \ln\left(\frac{[\text{Cl}^-]_o}{[\text{Cl}^-]_i}\right) = \frac{RT}{F} \ln\left(\frac{[\text{Cl}^-]_i}{[\text{Cl}^-]_o}\right).$$

(a)

Two weeks old: given $[\text{Cl}^-]_i = 7 \text{ mM}$

$$= -25 \ln(140/7)$$

$$= -25 \ln 20$$

$$\approx -74.86 \text{ mV}$$

For the newborn pup the reported equilibrium potential is $E_{\text{Cl}} \approx -40.24 \text{ mV}$. Solve for $[\text{Cl}^-]_i$:

$$-40.24 \text{ mV} = -25 \text{ mV} \cdot \ln\left(\frac{140}{[\text{Cl}^-]_i}\right) \Rightarrow \ln\left(\frac{140}{[\text{Cl}^-]_i}\right) = \frac{40.24}{25} = 1.6096.$$

Therefore

$$\frac{140}{[\text{Cl}^-]_i} = e^{1.6096} \approx 5.00 \Rightarrow [\text{Cl}^-]_i \approx \frac{140}{5.00} = 28 \text{ mM}.$$

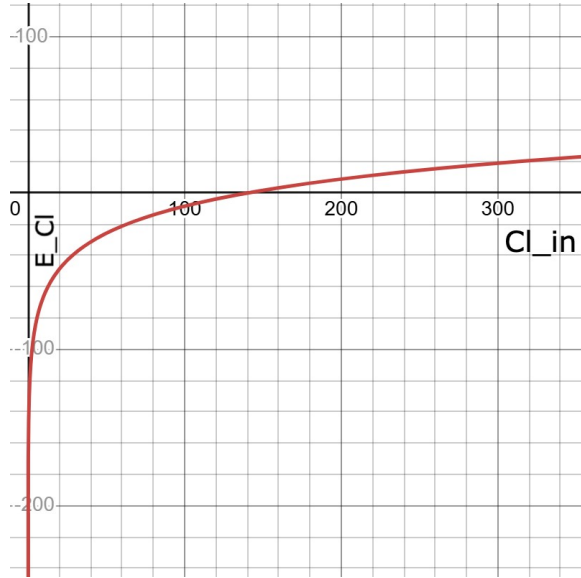
Answer (a): $[\text{Cl}^-]_i \approx 28 \text{ mM}$ (newborn hippocampal neurons).

(b) **Plot of E_{Cl} vs. $[\text{Cl}^-]_i$**

We plot

$$E_{\text{Cl}}([\text{Cl}^-]_i) = -25 \ln\left(\frac{140}{[\text{Cl}^-]_i}\right) \text{ mV},$$

for $[\text{Cl}^-]_i$ from (near) 0 to 140 mM.]



(c) Which hypothesis is consistent?

Hypotheses:

Observations: $[Cl^-]_i$ is greater at birth rather than two weeks later. This relationship is consistent with hypothesis 1 in which the change in intracellular concentration of chloride is due to a delayed expression of a chloride exporter.

2. GHK voltage for Na and K

The formula given:

$$E_{ghk} = \frac{RT}{F} \ln \frac{r_{Na}[Na^+]_o + (1 - r_{Na})[K^+]_o}{r_{Na}[Na^+]_i + (1 - r_{Na})[K^+]_i}, \quad r_{Na} = \frac{P_{Na}}{P_{Na} + P_K}.$$

(a) Range of r_{Na}

Since $r_{Na} = P_{Na}/(P_{Na} + P_K)$ and $P_{Na}, P_K \geq 0$, we have

$$0 \leq r_{Na} \leq 1.$$

(Endpoints: $r_{Na} = 0$ means pure K-permeability; $r_{Na} = 1$ means pure Na-permeability.)

(b) Monotonicity of E_{ghk} with respect to r_{Na}

Physically as r_{Na} increases, the membrane becomes more selective to Na. This means that the GHK potential moves from E_K towards E_{Na} . This is due to the fact that $r_{\text{Na}} = 0$ means pure K-permeability; $r_{\text{Na}} = 1$ means pure Na-permeability.

3. Combine three passive conductances into an effective leak

Step-by-step derivation

Start from the given current-balance equation

$$C \frac{dV}{dt} = I_{\text{app}} - g_K(V - E_K) - g_{\text{Na}}(V - E_{\text{Na}}) - g_{\text{Cl}}(V - E_{\text{Cl}}).$$

1. Expand each conductance term to separate the parts proportional to V and the constant terms:

$$\begin{aligned} -g_K(V - E_K) &= -g_K V + g_K E_K, \\ -g_{\text{Na}}(V - E_{\text{Na}}) &= -g_{\text{Na}} V + g_{\text{Na}} E_{\text{Na}}, \\ -g_{\text{Cl}}(V - E_{\text{Cl}}) &= -g_{\text{Cl}} V + g_{\text{Cl}} E_{\text{Cl}}. \end{aligned}$$

2. Substitute these expanded forms back into the differential equation:

$$C \frac{dV}{dt} = I_{\text{app}} - (g_K V + g_{\text{Na}} V + g_{\text{Cl}} V) + (g_K E_K + g_{\text{Na}} E_{\text{Na}} + g_{\text{Cl}} E_{\text{Cl}}).$$

3. Factor the terms proportional to V and collect the constant terms:

$$C \frac{dV}{dt} = I_{\text{app}} - (g_K + g_{\text{Na}} + g_{\text{Cl}})V + (g_K E_K + g_{\text{Na}} E_{\text{Na}} + g_{\text{Cl}} E_{\text{Cl}}).$$

4. Define the total (effective) leak conductance g_L as the sum of the individual conductances:

$$g_L \equiv g_K + g_{\text{Na}} + g_{\text{Cl}}.$$

With this, the equation becomes

$$C \frac{dV}{dt} = I_{\text{app}} - g_L V + (g_K E_K + g_{\text{Na}} E_{\text{Na}} + g_{\text{Cl}} E_{\text{Cl}}).$$

5. We want to write the right-hand side in the form $I_{\text{app}} - g_L(V - E_L)$.
Expand that target form:

$$I_{\text{app}} - g_L(V - E_L) = I_{\text{app}} - g_L V + g_L E_L.$$

Compare this with the previous expression. The terms $-g_L V$ already match; therefore the constant terms must match:

$$g_L E_L = g_K E_K + g_{\text{Na}} E_{\text{Na}} + g_{\text{Cl}} E_{\text{Cl}}.$$

6. Solve for the effective leak reversal potential E_L :

$$E_L = \frac{g_K E_K + g_{\text{Na}} E_{\text{Na}} + g_{\text{Cl}} E_{\text{Cl}}}{g_L} = \frac{g_K E_K + g_{\text{Na}} E_{\text{Na}} + g_{\text{Cl}} E_{\text{Cl}}}{g_K + g_{\text{Na}} + g_{\text{Cl}}}.$$

Final boxed results:

$$\boxed{g_L = g_K + g_{\text{Na}} + g_{\text{Cl}}},$$

$$\boxed{E_L = \frac{g_K E_K + g_{\text{Na}} E_{\text{Na}} + g_{\text{Cl}} E_{\text{Cl}}}{g_K + g_{\text{Na}} + g_{\text{Cl}}}}.$$

4. Two-conductance membrane

We consider

$$C \frac{dV}{dt} = I_{\text{app}} - g_K(V - E_K) - g_{\text{Na}}(V - E_{\text{Na}}).$$

(a)

If $I_{\text{app}} < 0$ and the steady-state voltage satisfies $V_{\infty} < E_K$, then for the potassium term

$$I_K = g_K(V - E_K).$$

When $V < E_K$ we have $V - E_K < 0$, hence $I_K < 0$, so the potassium current is **inward**. (Convention: positive current is outward.)

(b)

If $I_{\text{app}} > 0$ then a positive injected current tends to *depolarize* the membrane. So the applied current is **depolarizing**.

(c) Membrane time constant

Linearizing the right-hand side about a working point gives an effective total conductance $g_{\text{tot}} = g_K + g_{\text{Na}}$. The membrane time constant is

$$\tau = \frac{C}{g_{\text{tot}}} = \frac{C}{g_K + g_{\text{Na}}}.$$

(d) Steady-state potential for $g_K = 10 g_{\text{Na}}$ and $I_{\text{app}} = 0$

At steady-state ($dV/dt = 0$ and $I_{\text{app}} = 0$),

$$0 = -g_K(V - E_K) - g_{\text{Na}}(V - E_{\text{Na}}) \Rightarrow (g_K + g_{\text{Na}})V = g_K E_K + g_{\text{Na}} E_{\text{Na}}.$$

Let $g_{\text{Na}} = g$ and $g_K = 10g$. Then

$$V = \frac{10g \cdot E_K + g \cdot E_{\text{Na}}}{11g} = \frac{10E_K + E_{\text{Na}}}{11}.$$

With $E_K = -90$ mV and $E_{\text{Na}} = 60$ mV,

$$V = \frac{10(-90) + 60}{11} = \frac{-900 + 60}{11} = \frac{-840}{11} \approx -76.36 \text{ mV}.$$

Answer (d): $V \approx -76.4$ mV.

(e) Currents when $E_K < V < E_{\text{Na}}$

For $I_K = g_K(V - E_K)$: since $V > E_K$ the bracket is positive so $I_K > 0$ (potassium current is **outward**).

For $I_{\text{Na}} = g_{\text{Na}}(V - E_{\text{Na}})$: since $V < E_{\text{Na}}$ the bracket is negative so $I_{\text{Na}} < 0$ (sodium current is **inward**).

Problem 5

A cell membrane has $R_m = 20$ M Ω and $C_m = 120$ pF. The initial membrane potential is $V(0) = -100$ mV and the resting potential is $V_\infty = -60$ mV.

(a) Membrane conductance

$$g_m = \frac{1}{R_m} = \frac{1}{20 \times 10^6 \Omega} = 5.0 \times 10^{-8} \text{ S}$$

(b) Membrane time constant

$$\tau_m = R_m C_m = (20 \times 10^6 \Omega)(120 \times 10^{-12} \text{ F}) = 2.4 \times 10^{-3} \text{ s} = 2.4 \text{ ms}$$

(c) Membrane potential after 10 ms

General charging equation:

$$V(t) = V_\infty + (V(0) - V_\infty)e^{-t/\tau_m}$$

Substitute:

$$\begin{aligned} V(10 \text{ ms}) &= -60 + (-100 - (-60))e^{-10/2.4} \\ &= -60 - 40e^{-4.17} \approx -60 - 40(0.0153) = -60.61 \text{ mV} \end{aligned}$$

Thus, the potential is more depolarized than the initial -100 mV .

(d) Time to reach -70 mV

$$\begin{aligned} -70 &= -60 + (-100 + 60)e^{-t/\tau_m} \\ -70 &= -60 - 40e^{-t/2.4} \\ -10 &= -40e^{-t/2.4} \\ e^{-t/2.4} &= 0.25 \quad \Rightarrow \quad -\frac{t}{2.4} = \ln(0.25) \\ t &= 2.4 \ln(4) \approx 3.33 \text{ ms} \end{aligned}$$

(e) Surface area of the cell

Specific capacitance:

$$c_m = 1 \mu\text{F}/\text{cm}^2 = 1 \times 10^{-6} \text{ F}/10^{-4} \text{ m}^2 = 0.01 \text{ F}/\text{m}^2$$

Surface area:

$$A = \frac{C_m}{c_m} = \frac{120 \times 10^{-12}}{0.01} = 1.2 \times 10^{-8} \text{ m}^2$$

Convert to μm^2 :

$$A = 1.2 \times 10^{-8} \times (10^{12}) = 1.2 \times 10^4 \mu\text{m}^2$$

Problem 6

A spherical cell of radius $r = 5 \mu\text{m}$ experiences a 1 ms inward sodium current of $1 \text{ mA}/\text{cm}^2$.

Step 1: Total surface area

$$A = 4\pi r^2 = 4\pi(5 \times 10^{-6})^2 = 3.14 \times 10^{-10} \text{ m}^2$$

Step 2: Volume of the cell

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(5 \times 10^{-6})^3 = 5.24 \times 10^{-16} \text{ m}^3 = 5.24 \times 10^{-13} \text{ L}$$

Step 2: Current density

$$J = 1 \text{ mA}/\text{cm}^2 = \frac{10^{-3} \text{ A}}{10^{-4} \text{ m}^2} = 10 \text{ A}/\text{m}^2$$

Step 3: Charge entry

$$\text{Current} = \frac{\text{Charge}}{\text{Time}}$$

$$\text{Charge} = \text{Time} * \text{Current}$$

$$Q = I\Delta t = (3.14 \times 10^{-9})(1 \times 10^{-3}) = 3.14 \times 10^{-12} \text{ C}$$

Step 4: Number of Na^+ ions

$$N = \frac{\text{Charge}}{\text{Faraday's Constant}}$$

This is because Faraday's constant is $\frac{\text{Coulombs}}{\text{Moles}}$

$$N = \frac{Q}{e} = \frac{3.14 \times 10^{-12}}{96485} \approx 3.25 \times 10^{-17}$$

Step 5: Concentration

$$\Delta[\text{Na}^+] = \frac{n}{V} = \frac{3.25 \times 10^{-17}}{5.24 \times 10^{-13}} = 6.2 \times 10^{-5} \text{ M} = 62 \mu\text{M}$$

Step 6: Percentage increase

Assume baseline $[\text{Na}^+]_i = 10 \text{ mM}$:

$$\% \Delta = \frac{62 \mu\text{M}}{10 \text{ mM}} \times 100 = 0.62\%$$