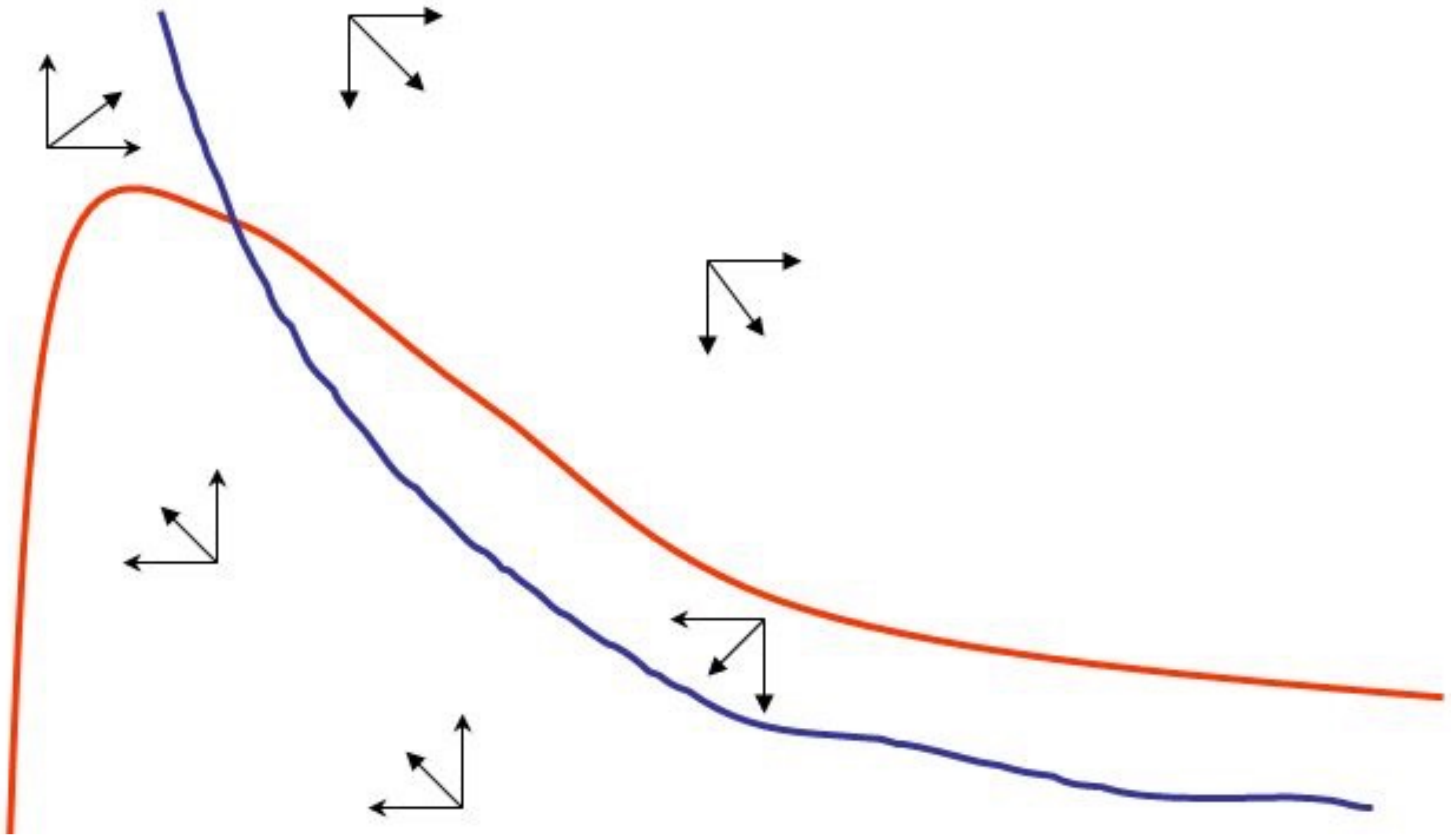
Cellular Biophysics & Modeling

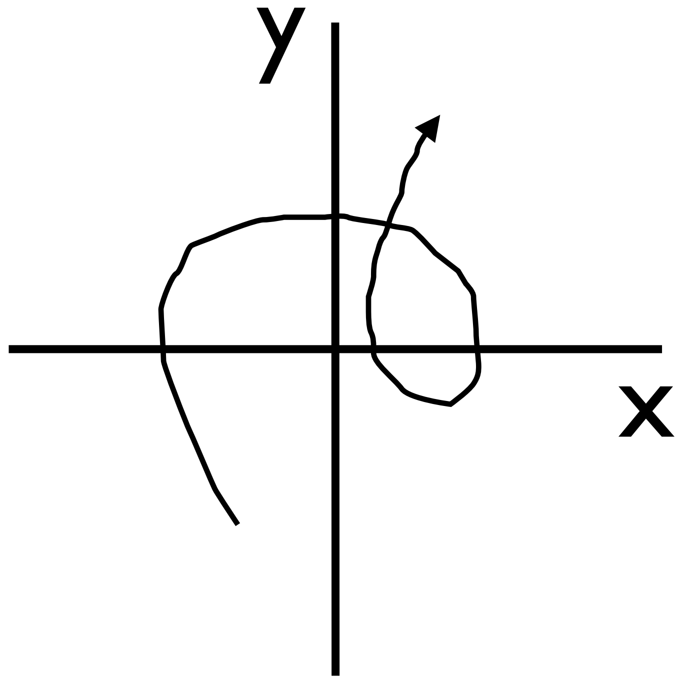
Lecture 11

phase plane analysis of 2D nonlinear systems



The "phase plane" for 2D autonomous ODEs

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

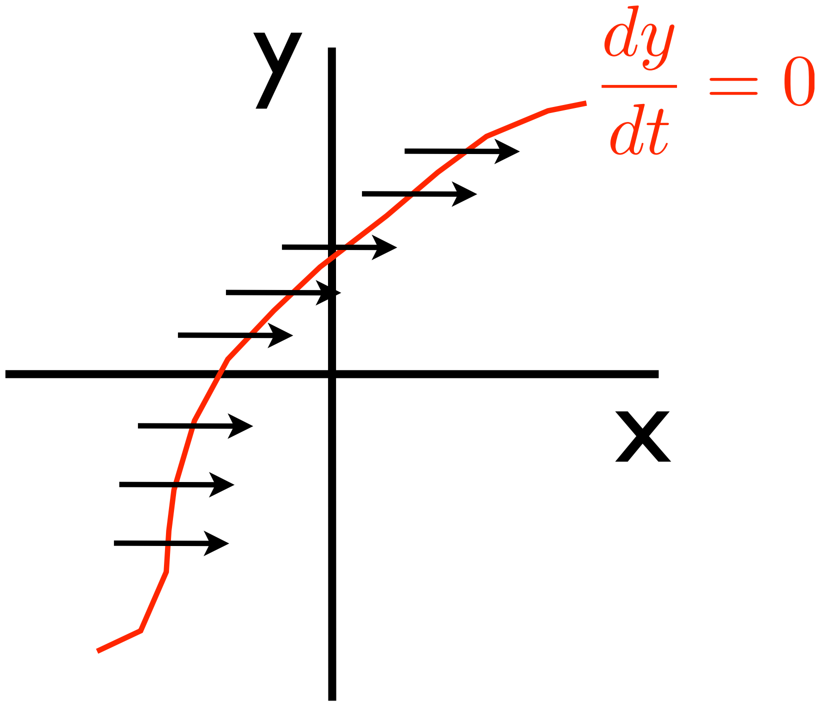


Solutions of ODE system are curves in (x,y)-plane

Can a solution intersect itself? Why or why not?

“Nullclines” for 2D autonomous ODEs

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$



$$x \text{ nullcline: } \{(x,y) : 0 = f(x,y)\}$$
$$y \text{ nullcline: } \{(x,y) : 0 = g(x,y)\}$$

Intersections of nullclines are equilibria of 2D system

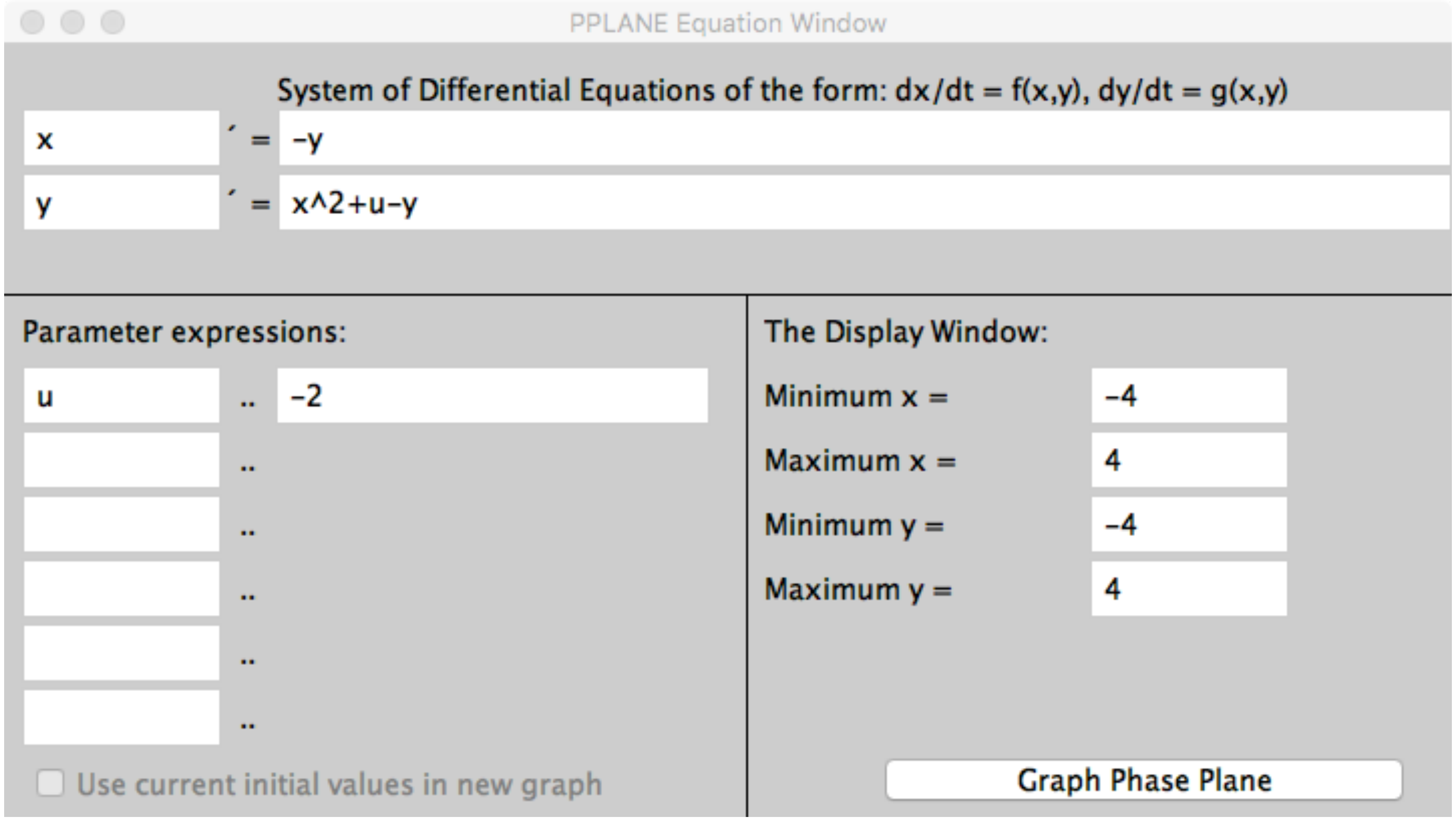
Flow must cross nullclines with rate either vertically or horizontally. Why?

Numerical phase plane analysis

$$\frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = x^2 - y + u$$

Try this yourself in PPLANE
(see Resources)
http://math.rice.edu/~dfield/dfpp.html



System of Differential Equations of the form: dx/dt = f(x,y), dy/dt = g(x,y)

x ′ = -y

y ′ = x^2+u-y

Parameter expressions:

u .. -2

The Display Window:

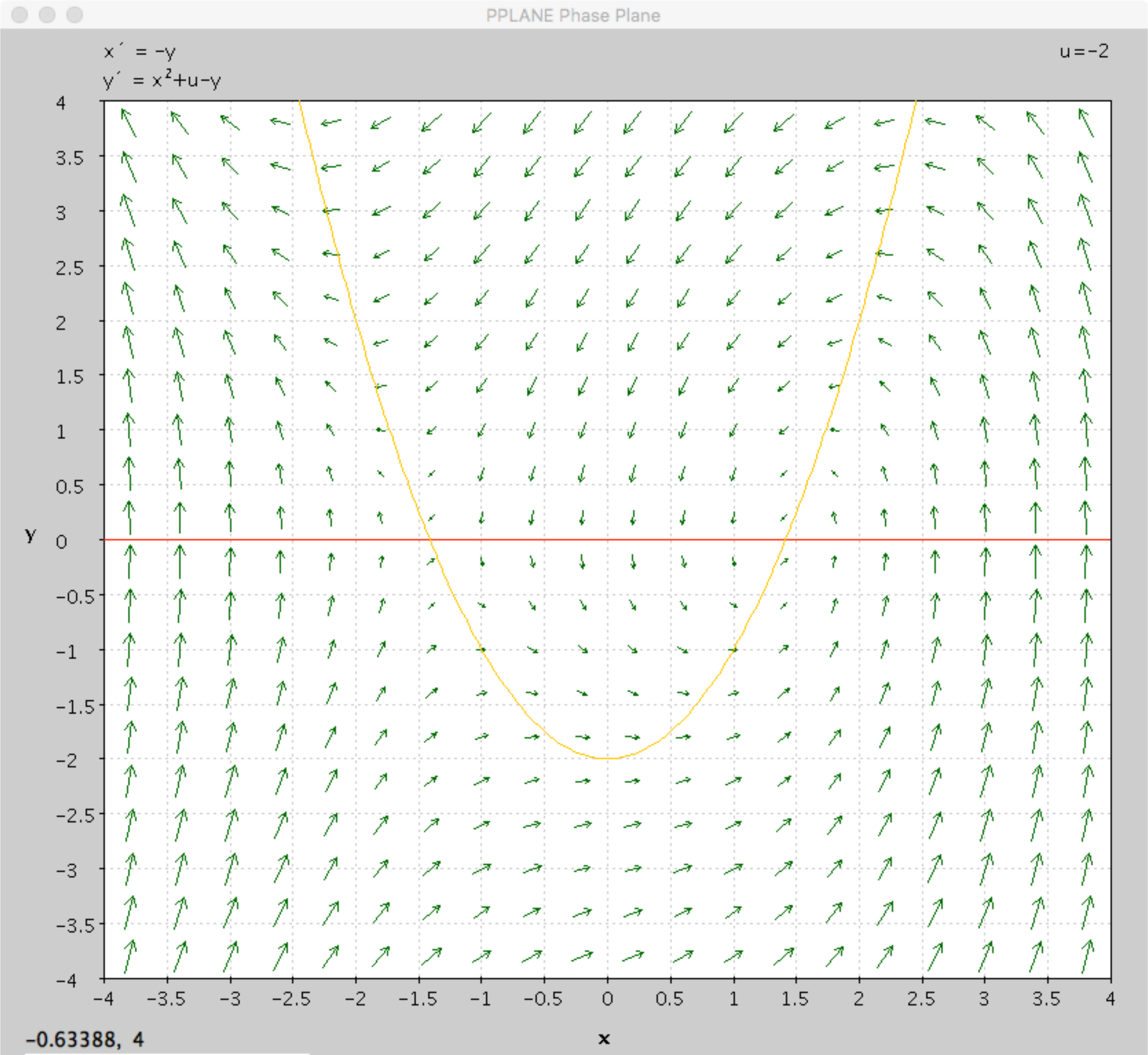
Minimum x = -4

Maximum x = 4

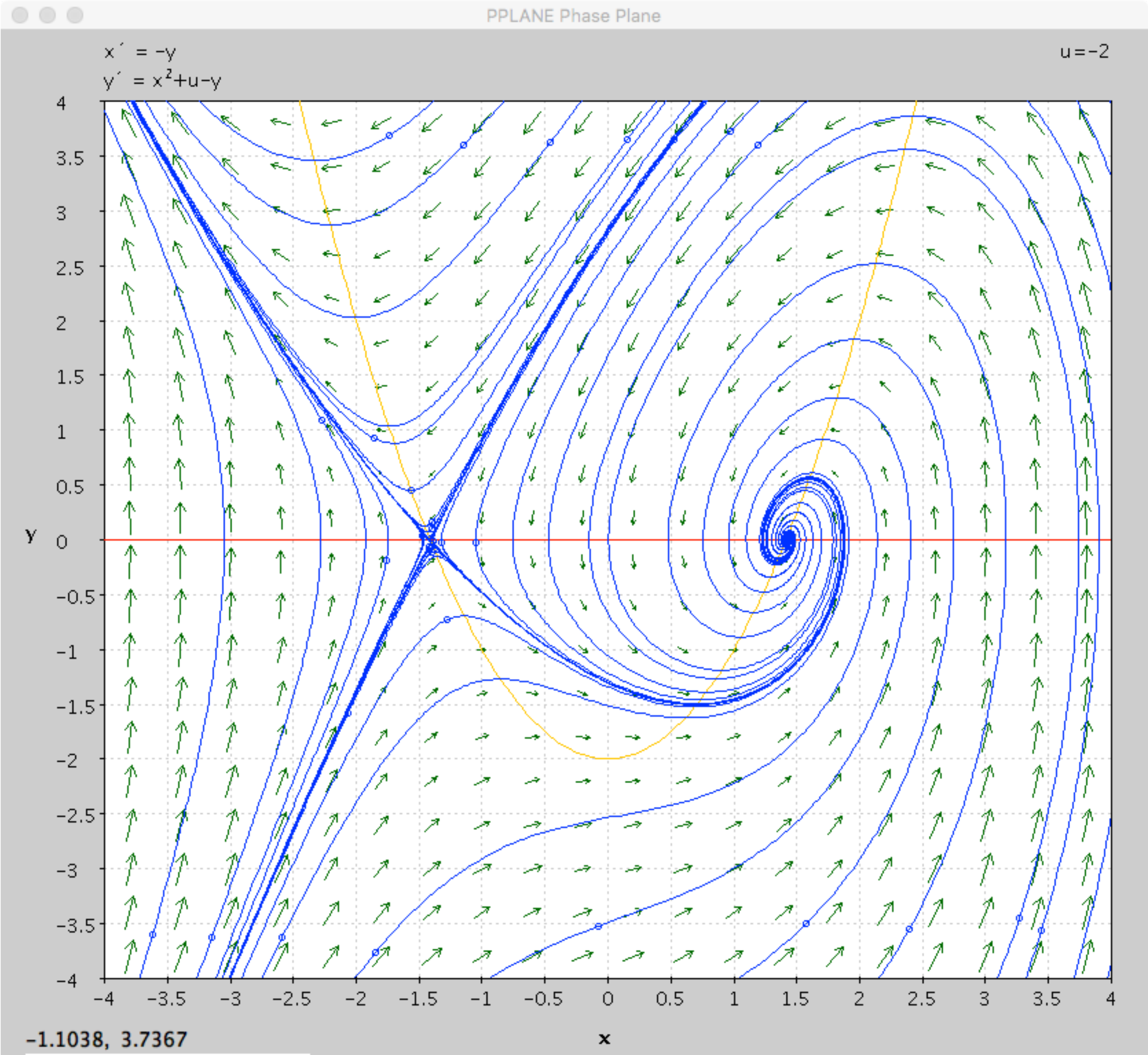
Minimum y = -4

Maximum y = 4

x-nullcline, y-nullcline and direction field



… and trajectories



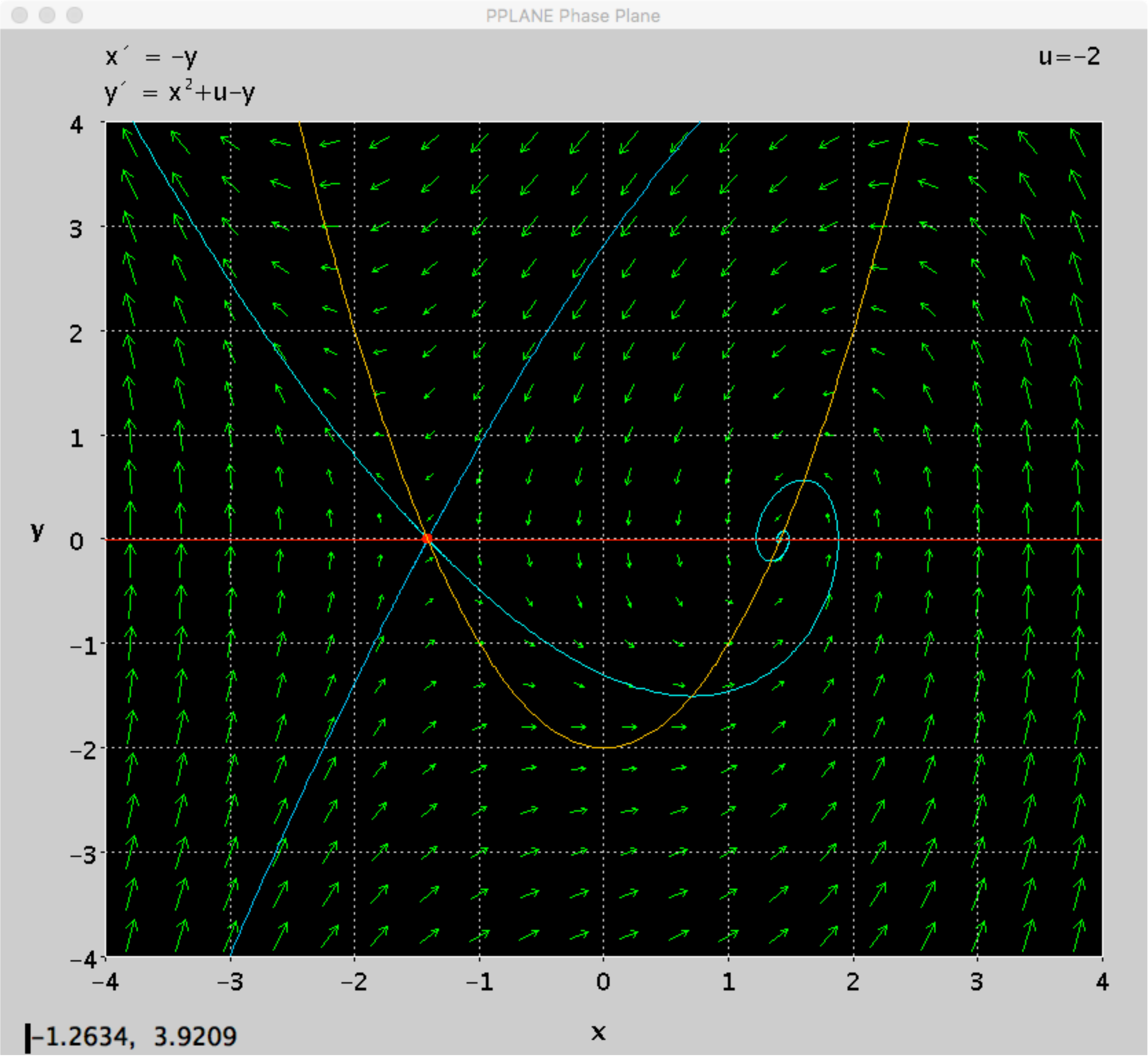
$x' = -y$

$y' = x^2+u-y$

u=-2

-1.1038, 3.7367

stable and unstable manifold of saddle



$x' = -y$

$y' = x^2+u-y$

u=−2

−1.2634, 3.9209

Analytical phase plane analysis

$$\frac{dx}{dy} = -y$$

x-nullcline: $0 = -y \Rightarrow y = 0$

$$\frac{dy}{dt} = x^2 - y + \mu$$

y-nullcline $0 = x^2 - y + \mu$

$y = x^2 + \mu$

steady-states

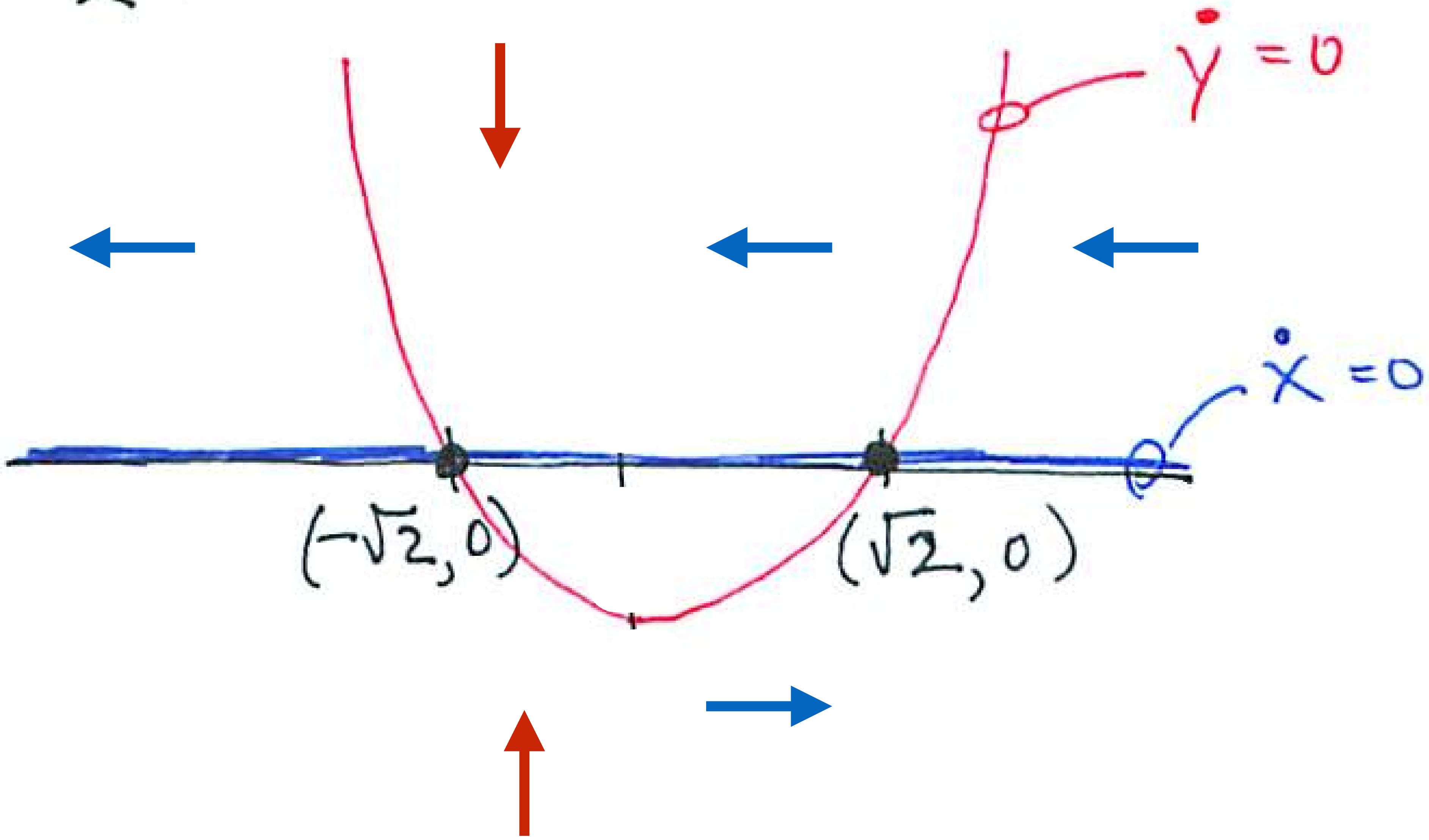
$$\begin{cases} y = 0 \\ y = x^2 + \mu \end{cases} \Rightarrow 0 = x^2 + \mu \Rightarrow x^2 = -\mu \Rightarrow x = \pm\sqrt{-\mu}$$

so... two steady states w/ $\mu < 0$ given by

one " " " $\mu = 0$ given by $x = 0$

zero " " " $\mu > 0$

Say $\mu = -2$:



Are the steady states stable or unstable?

In one dimension we calculated the derivative f'(y) and evaluated at steady-state. The sign of f'(yss) determined stability.

In two dimensions we linearize equations and evaluate the Jacobian matrix at each steady state. The eigenvalues of the Jacobian matrix determine stability. The eigenvalues are related to the trace and determinant of Jacobian, so steady state can be classified by locating tr(J) and det(J) on the trace-determinant plane.

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

in general, matrix entries are functions of x and y

$$J(x_{ss}, y_{ss}) = \left. \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \right|_{(x,y)=(x_{ss},y_{ss})}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

constant matrix

Near steady state dynamics are well-approximated by this 2D linear system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

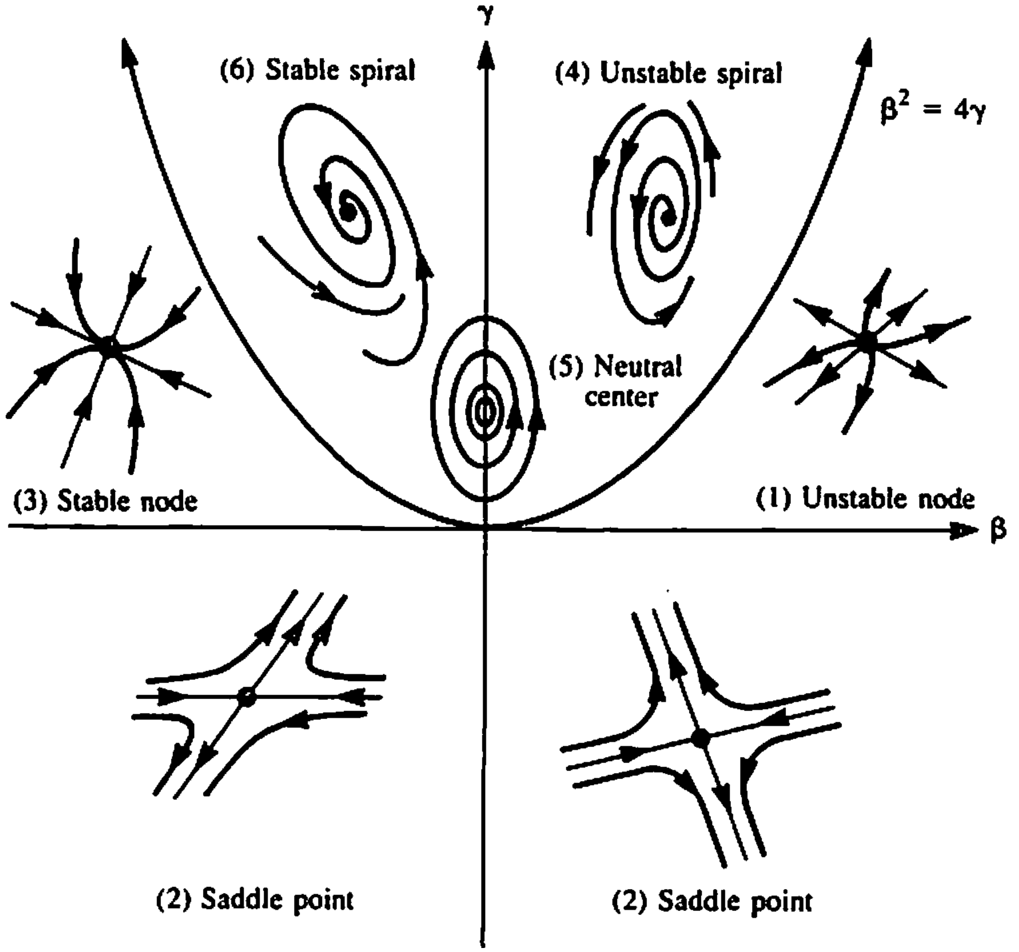
⟺

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

$$\beta = \mathrm{tr}(J) = a + d$$

$$\gamma = \det(J) = ad - bc$$

1. Unstable node: $\beta > 0$ and $\gamma > 0$.
2. Saddle point: $\gamma < 0$.
3. Stable node: $\beta < 0$ and $\gamma > 0$.
4. Unstable spiral: $\beta^2 < 4\gamma$ and $\beta > 0$.
5. Neutral center: $\beta^2 < 4\gamma$ and $\beta = 0$.
6. Stable spiral: $\beta^2 < 4\gamma$ and $\beta < 0$.



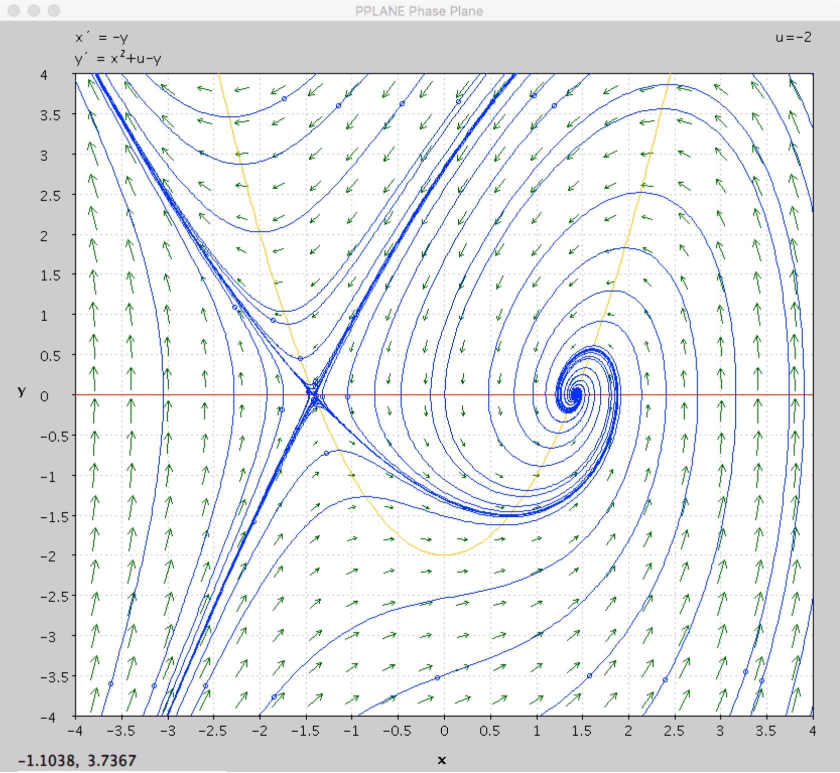
$$\frac{dx}{dy} = -y \qquad \frac{dy}{dt} = x^2 - y + \mu \quad \Longrightarrow \quad J = \begin{pmatrix} 0 & -1 \\ 2x & -1 \end{pmatrix}$$

What about stability of these steady states?

$(\sqrt{2}, 0)$:

$$J(\sqrt{2},0) = \begin{pmatrix} 0 & -1 \\ 2\sqrt{2} & -1 \end{pmatrix} \qquad \left.\begin{array}{l} \text{trace } J = -1 \\ \det J = 2\sqrt{2} \end{array}\right\} \Rightarrow \underline{\underline{\text{stable}}}!$$

this is a stable spiral because $\det J > (\text{trace } J)^2/4$
(see lecture slide with trace-determinant plane)



$(-\sqrt{2}, 0)$:

$$J(-\sqrt{2},0) = \begin{pmatrix} 0 & -1 \\ -2\sqrt{2} & -1 \end{pmatrix} \qquad \left.\begin{array}{l} \text{trace } J = -1 \\ \det J = -2\sqrt{2} \end{array}\right\} \Rightarrow \text{unstable}$$

the determinant is negative so this is a <u>saddle</u>.

This works because solutions of linear 2D system can be written in terms of eigenvectors and eigenvalues of the constant coefficient matrix

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

eigenvector

Assume:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} e^{\lambda t} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

eigenvalue

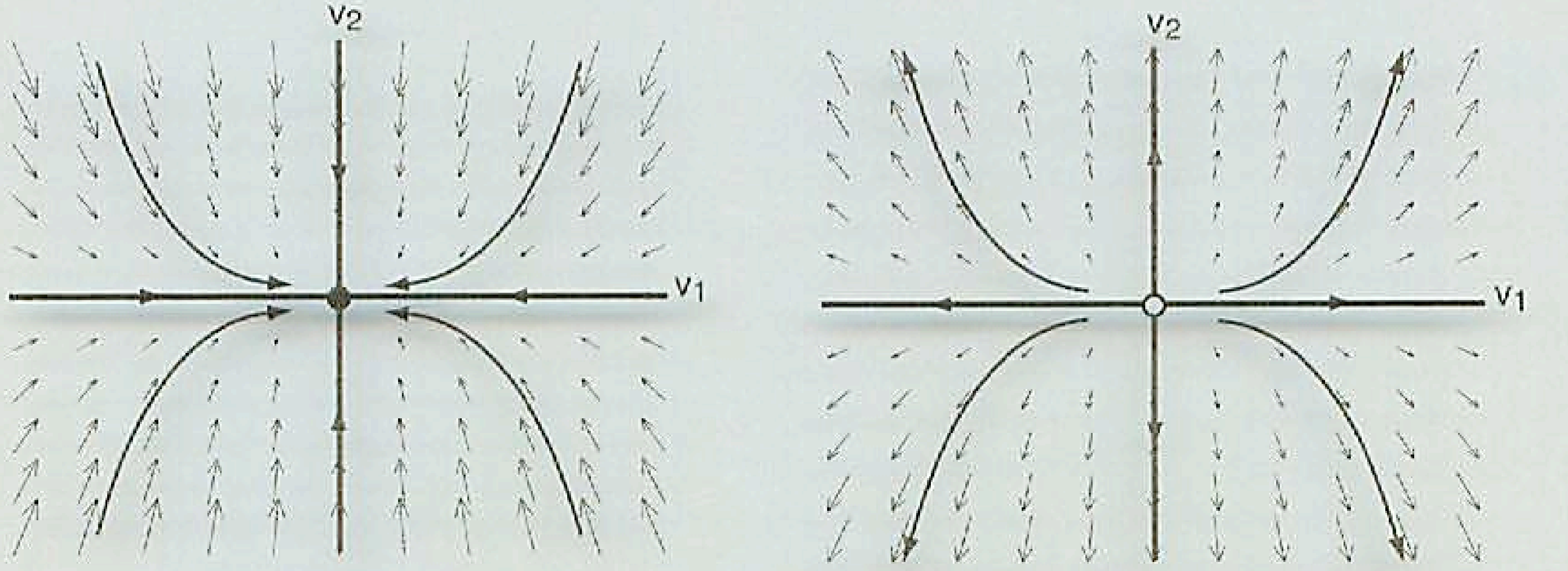
eigenvalues

$$\lambda = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$$

$$\beta = \mathrm{tr}(J) = a + d$$
$$\gamma = \det(J) = ad - bc$$

stable node or sink

unstable node or source



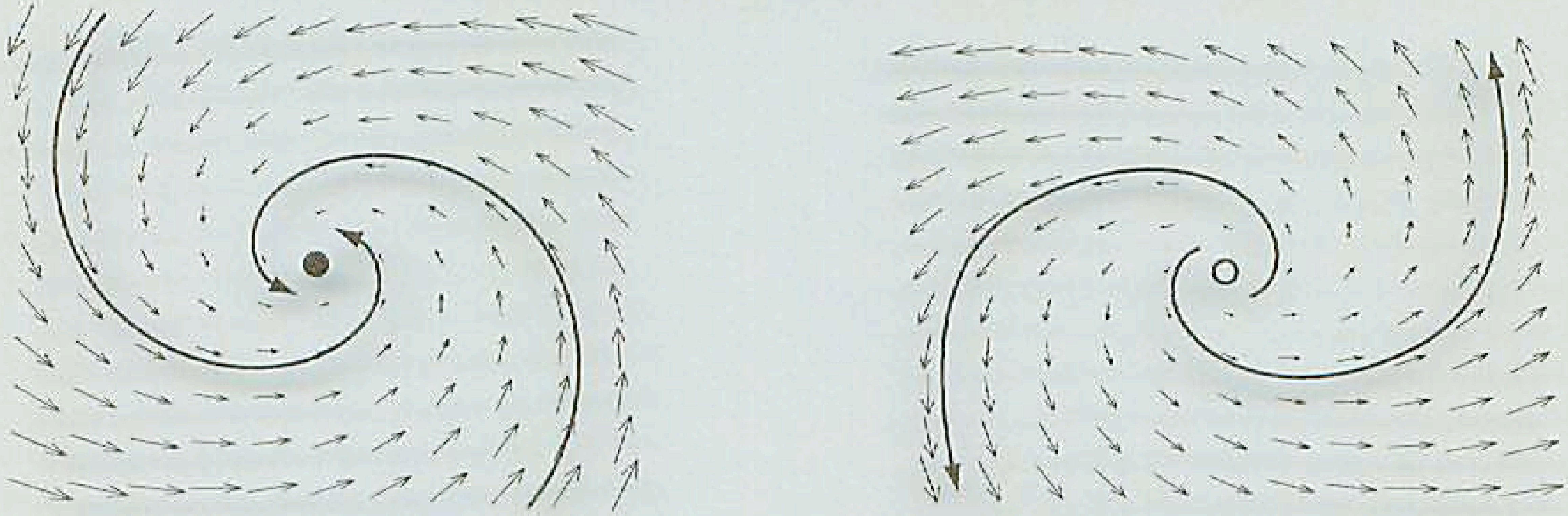
stable node

unstable node

FIGURE 4.19. Node equilibrium occurs when both eigenvalues are real and have the same sign, e.g., $\lambda_1 = -1$ and $\lambda_2 = -3$ (stable) or $\lambda_1 = +1$ and $\lambda_2 = +3$ (unstable). Most trajectories converge to or diverge from the node along the eigenvector v_1 corresponding to the eigenvalue having smallest absolute value.

stable spiral or focus

unstable spiral or focus



stable focus

unstable focus

FIGURE 4.21. Focus equilibrium occurs when eigenvalues are complex-conjugate, e.g., $\lambda = -3 \pm i$ (stable) or $\lambda = +3 \pm i$ (unstable). The imaginary part (here 1) determines the frequency of rotation around the focus.

saddle point (unstable)

lines in v1 and v2 directions are invariant sets

v1: unstable manifold
v2: stable manifold

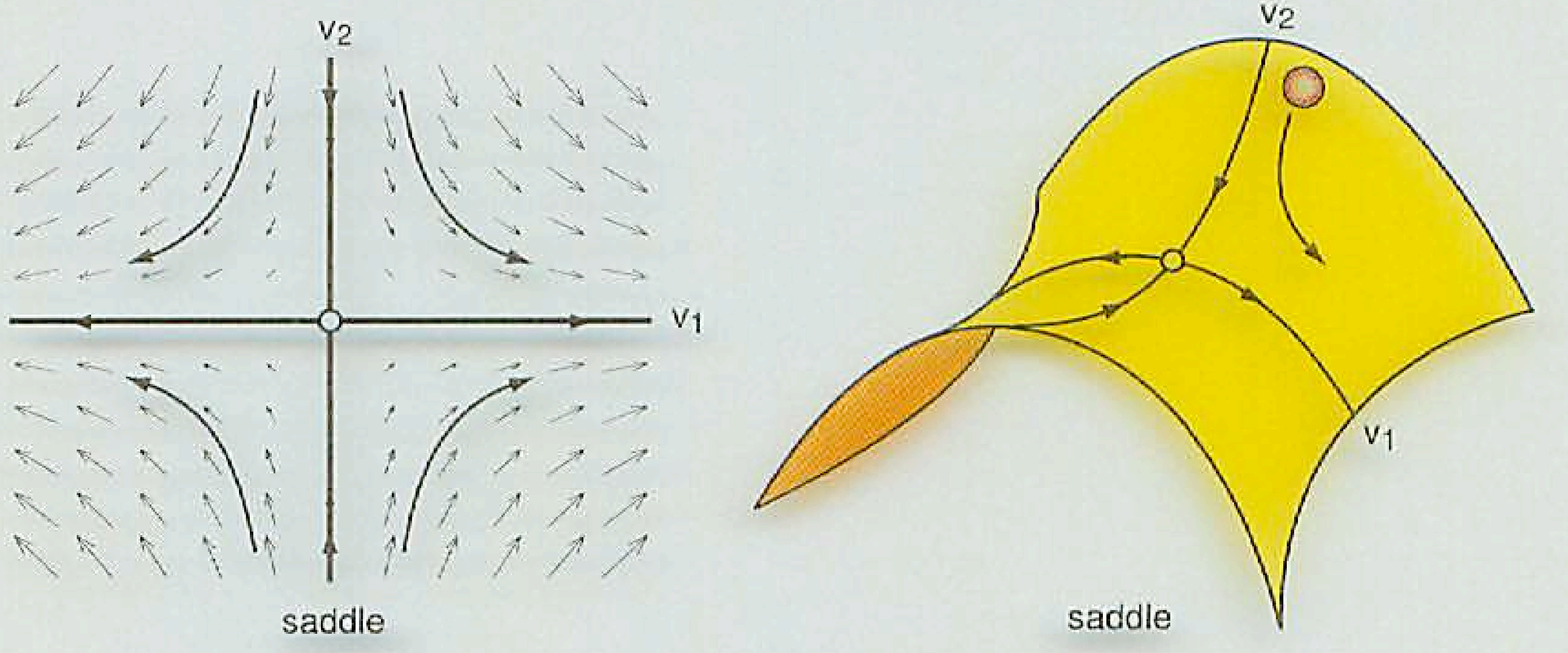
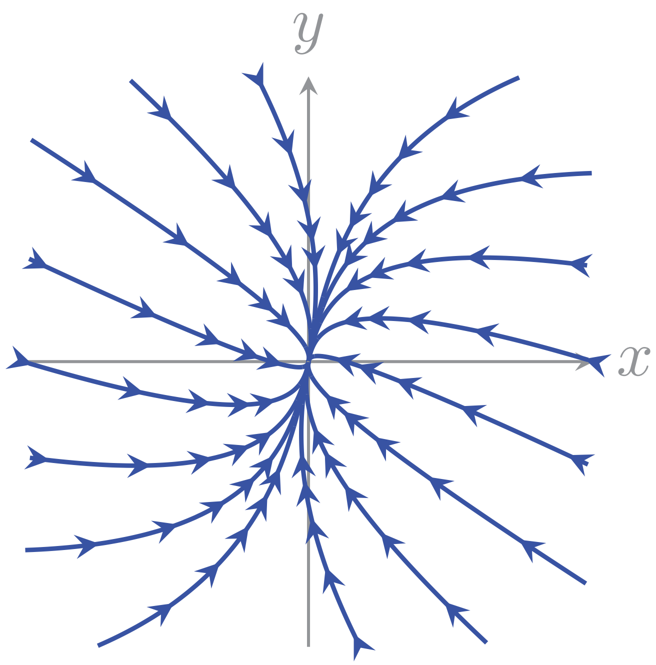
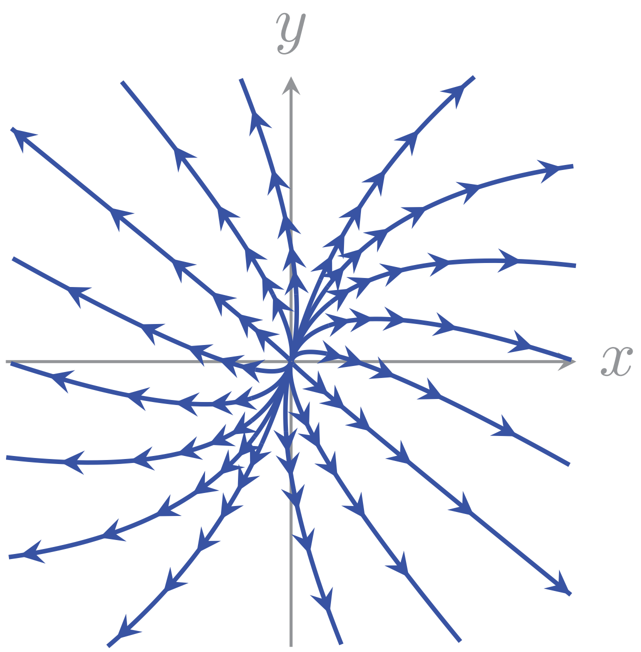


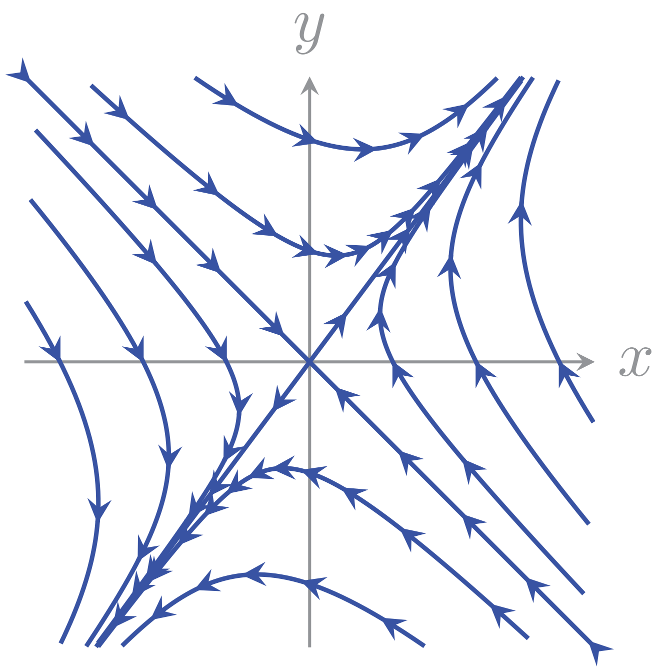
FIGURE 4.20. Saddle equilibrium occurs when two real eigenvalues have opposite signs, e.g., $\lambda_1 = +1$ and $\lambda_2 = -1$. Most trajectories diverge from the equilibrium along the eigenvector corresponding to the positive eigenvalue (in this case v_1).



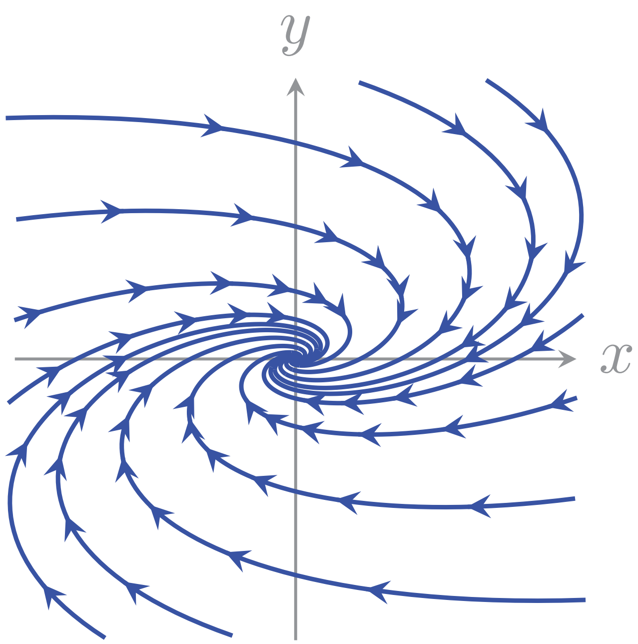
$\lambda_1 < \lambda_2 < 0$
stable node



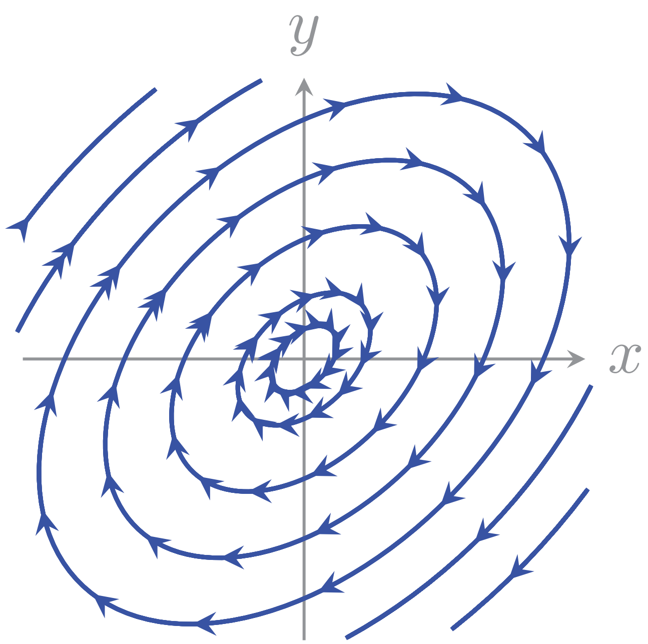
$0 < \lambda_1 < \lambda_2$
unstable node



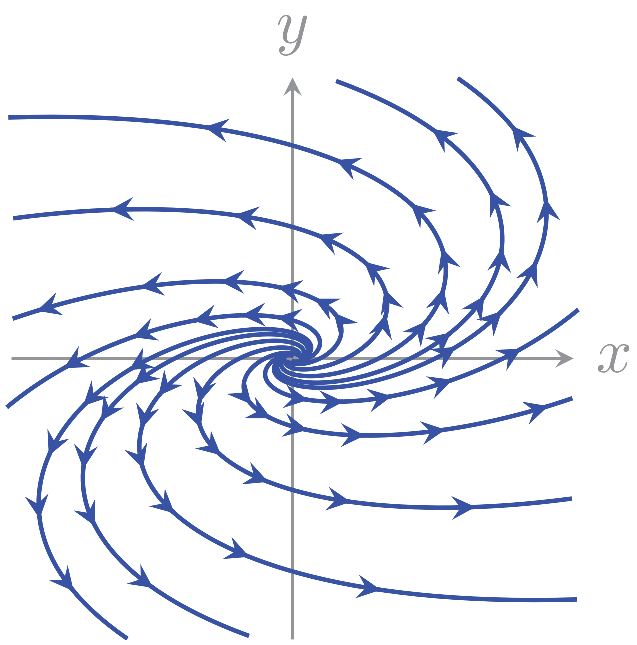
$\lambda_1 < 0 < \lambda_2$
saddle



$\lambda = \mu \pm i\omega$ with $\mu < 0$
stable spiral

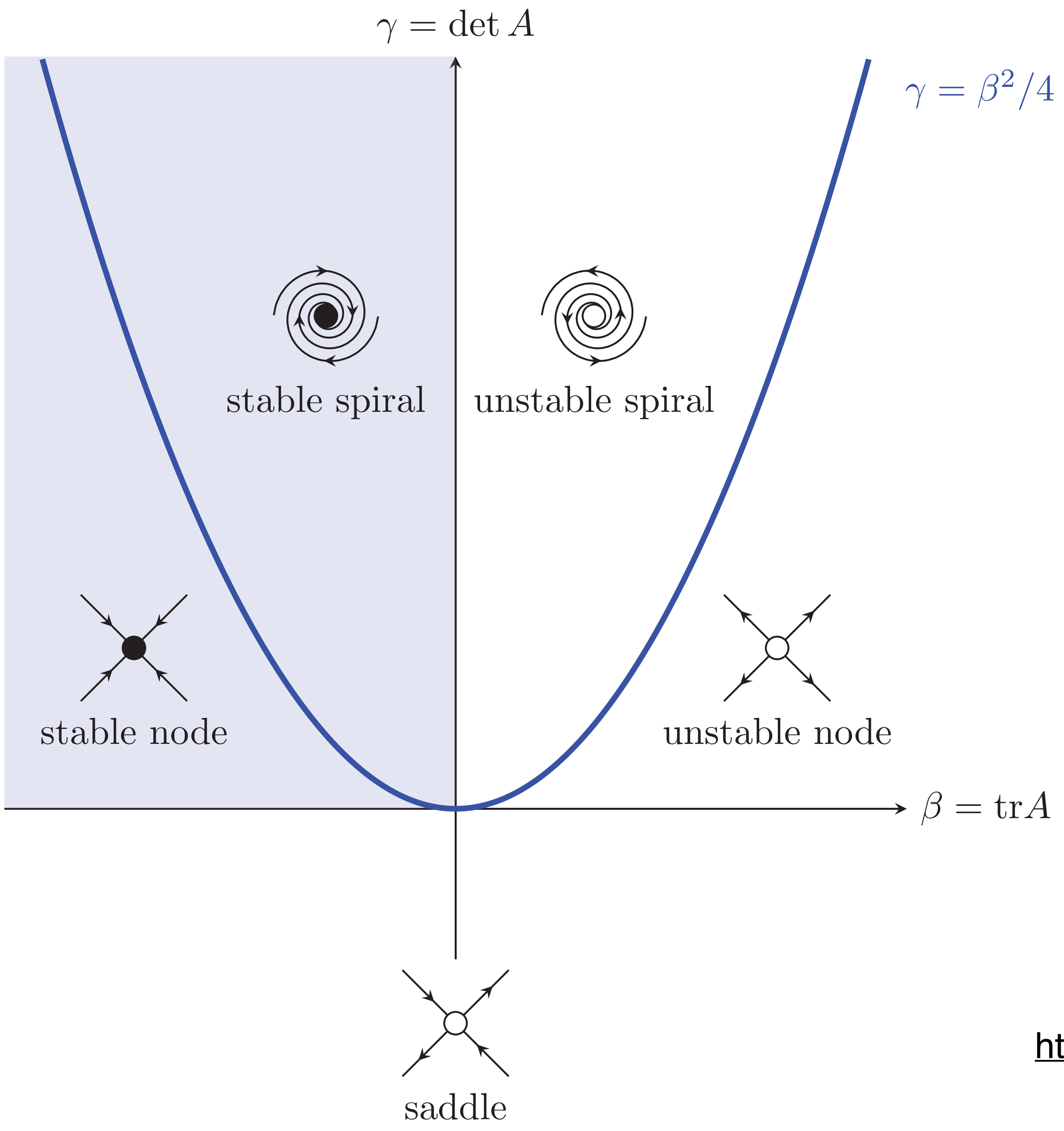


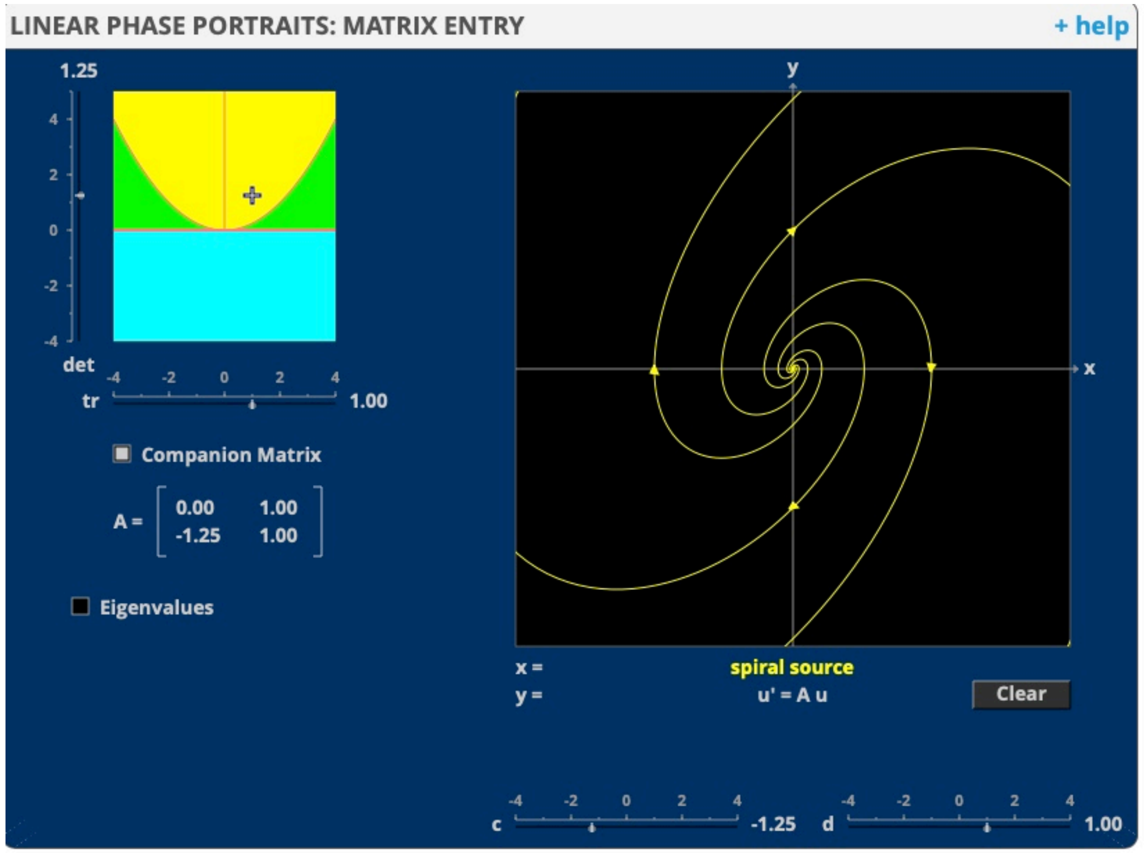
$\lambda = \pm i\omega$
center



$\lambda = \mu \pm i\omega$ with $\mu > 0$
unstable spiral

trace-determinant plane




LINEAR PHASE PORTRAITS: MATRIX ENTRY

det 1.25

tr 1.00

☐ Companion Matrix

$$A = \begin{bmatrix} 0.00 & 1.00 \\ -1.25 & 1.00 \end{bmatrix}$$

☐ Eigenvalues

spiral source

u' = A u

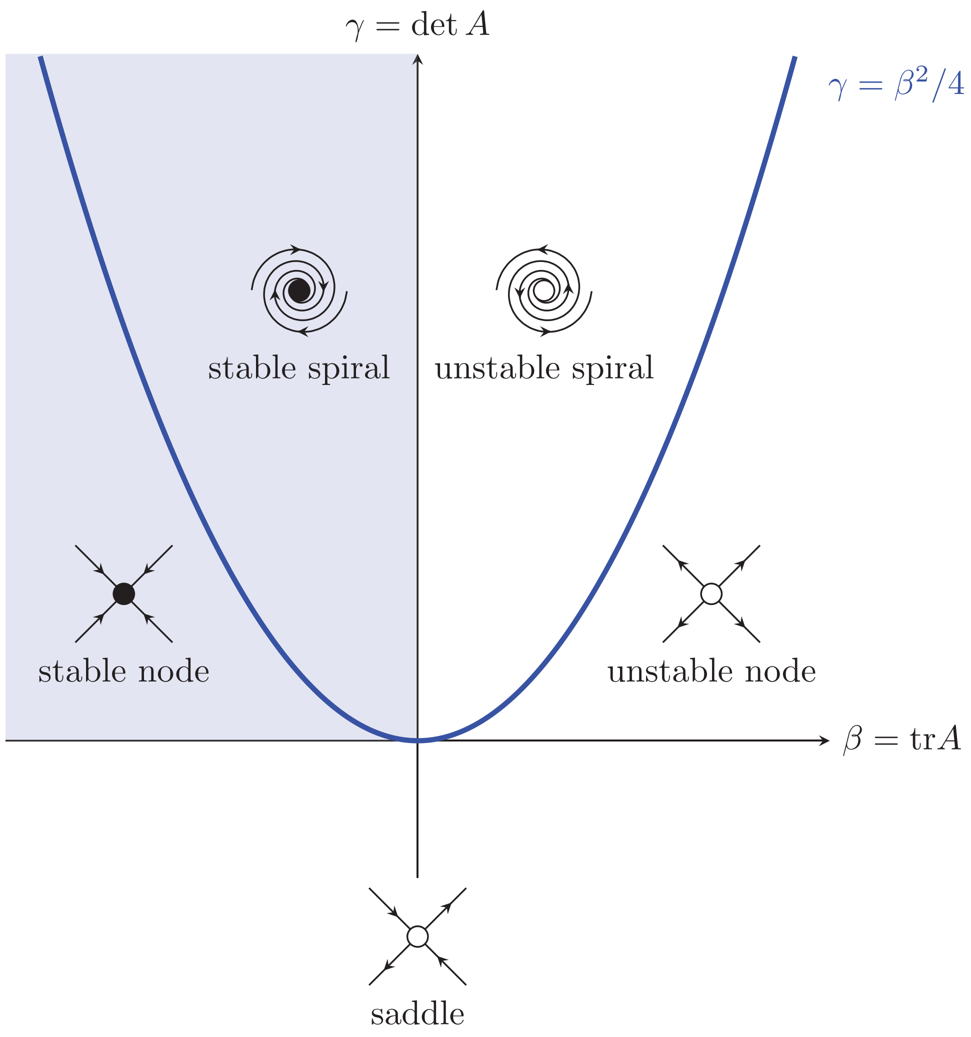
c -1.25

d 1.00

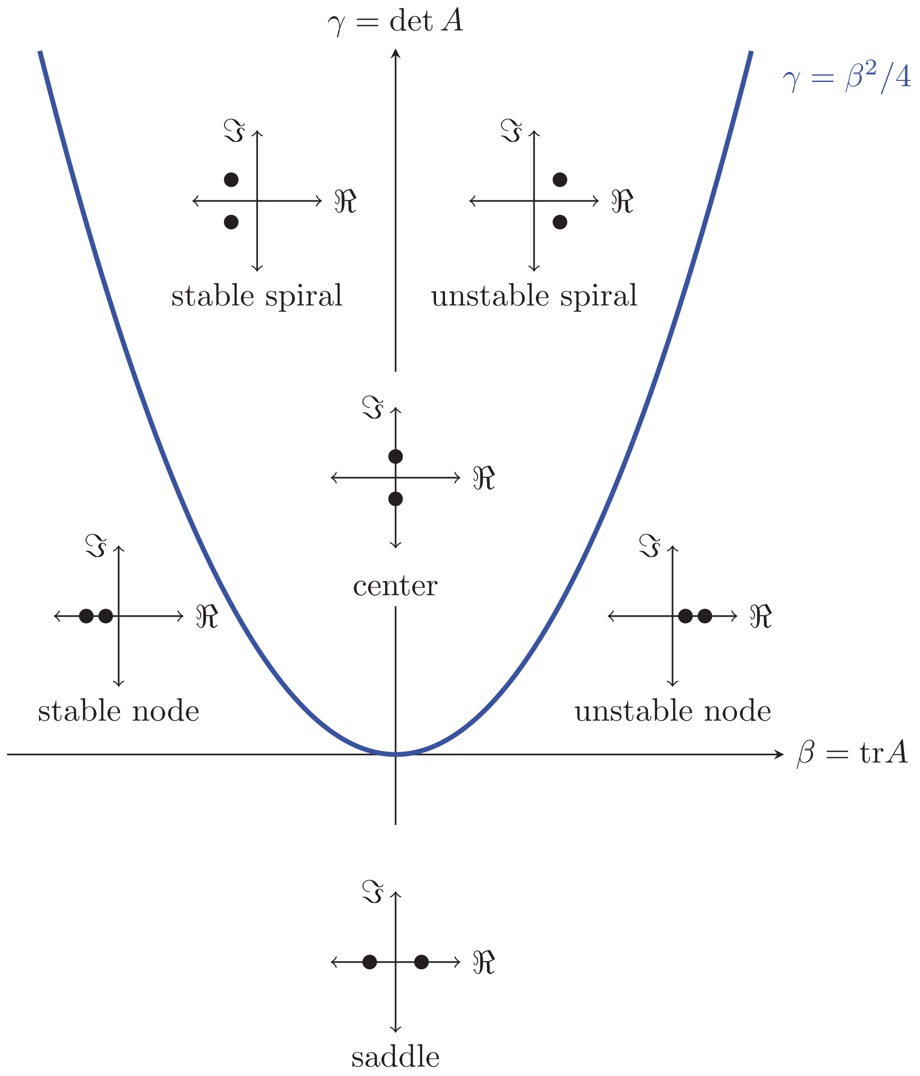
https://mathlets.org/mathlets/linear-phase-portraits-matrix-entry/

trace-determinant plane

classification



eigenvalues



Another example of phase plane analysis

$$\begin{cases} \dfrac{dx}{dt} = \overbrace{x - xy}^{f(x,y)} \\[2ex] \dfrac{dy}{dt} = \underbrace{-y + xy}_{g(x,y)} \end{cases}$$

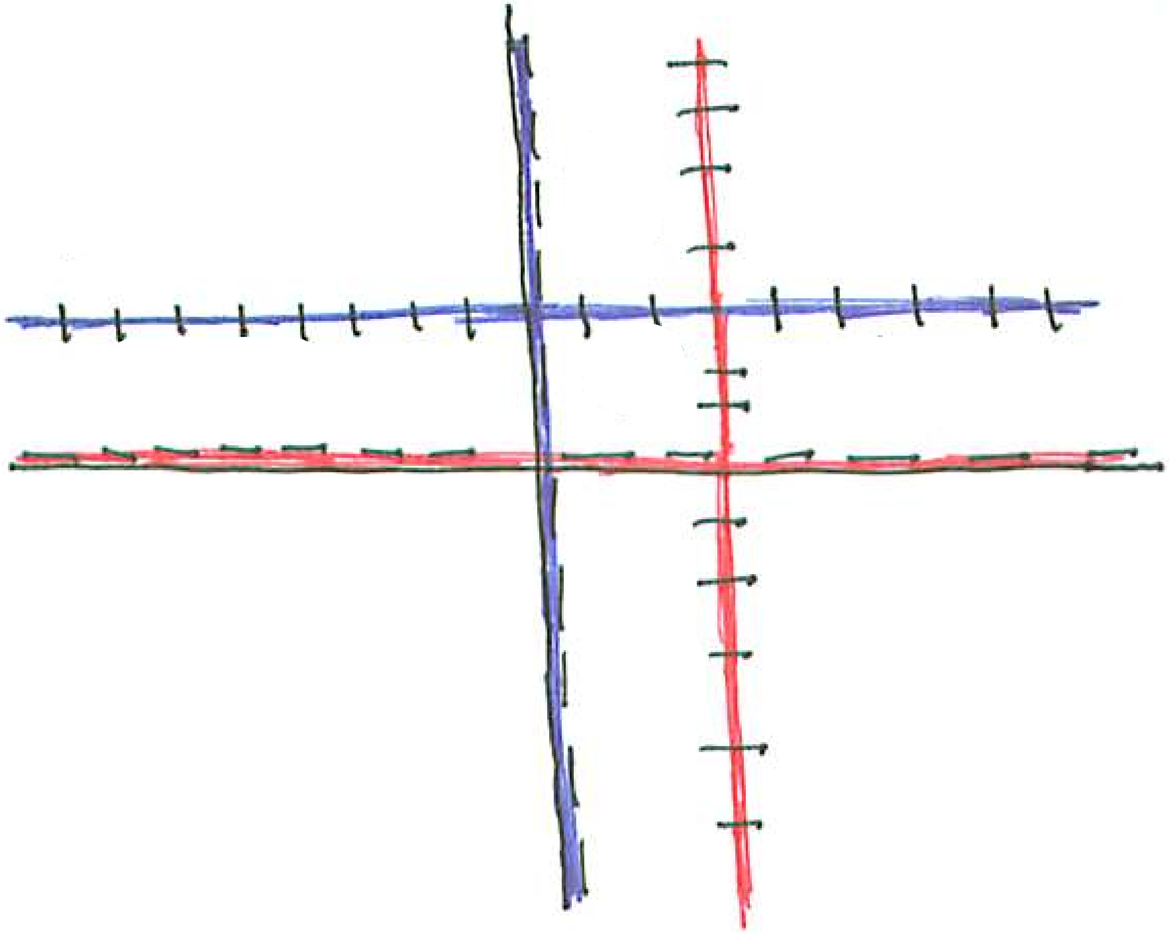
$$\begin{cases} \dfrac{dx}{dt} = \overbrace{x - xy}^{f(x,y)} \\ \dfrac{dy}{dt} = \underbrace{-y + xy}_{g(x,y)} \end{cases}$$

x-nullcline:

$$0 = x - xy$$
$$0 = x(1-y)$$
$$x = 0 \text{ or } y = 1$$

y-nullcline:

$$0 = -y + xy$$
$$0 = y(x-1)$$
$$y = 0 \text{ or } x = 1$$



steady states:

$$(x_{ss}, y_{ss}) = (0,0)$$

$$(x_{ss}, y_{ss}) = (1,1)$$

ⓐ rates of change in the region where $x<0$ and $y>1$ are

$$\begin{cases} \dfrac{dx}{dt} = \overbrace{x - xy}^{f(x,y)} \\[2ex] \dfrac{dy}{dt} = \underbrace{-y + xy}_{g(x,y)} \end{cases}$$

x-nullcline: $0 = x - xy$

$0 = x(1-y)$

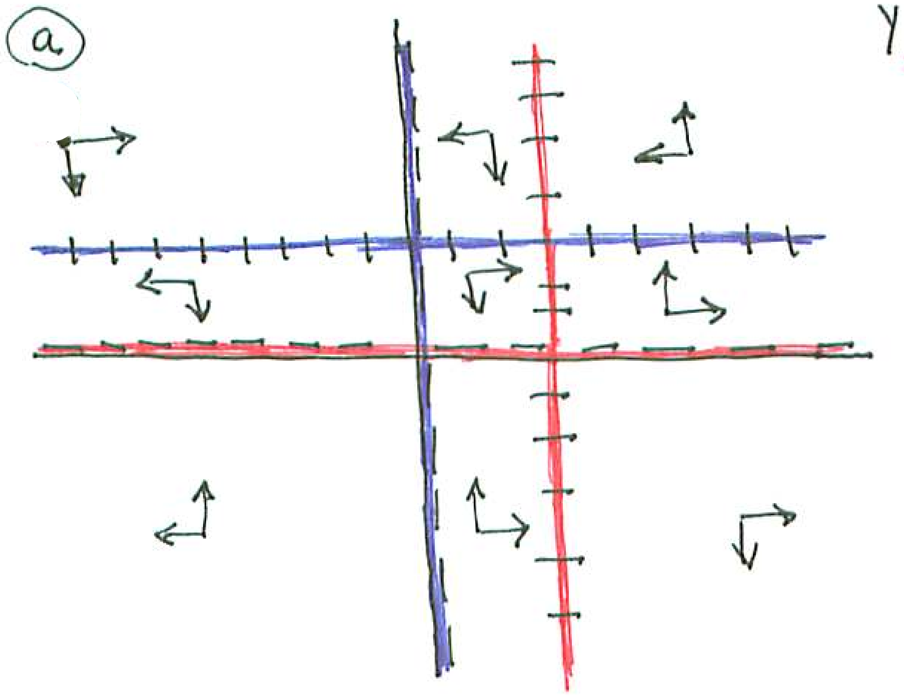
$x = 0$ or $y = 1$

y-nullcline: $0 = -y + xy$

$0 = y(x-1)$

$y = 0$ or $x = 1$

ⓐ



steady states:

$(x_{ss}, y_{ss}) = (0,0)$

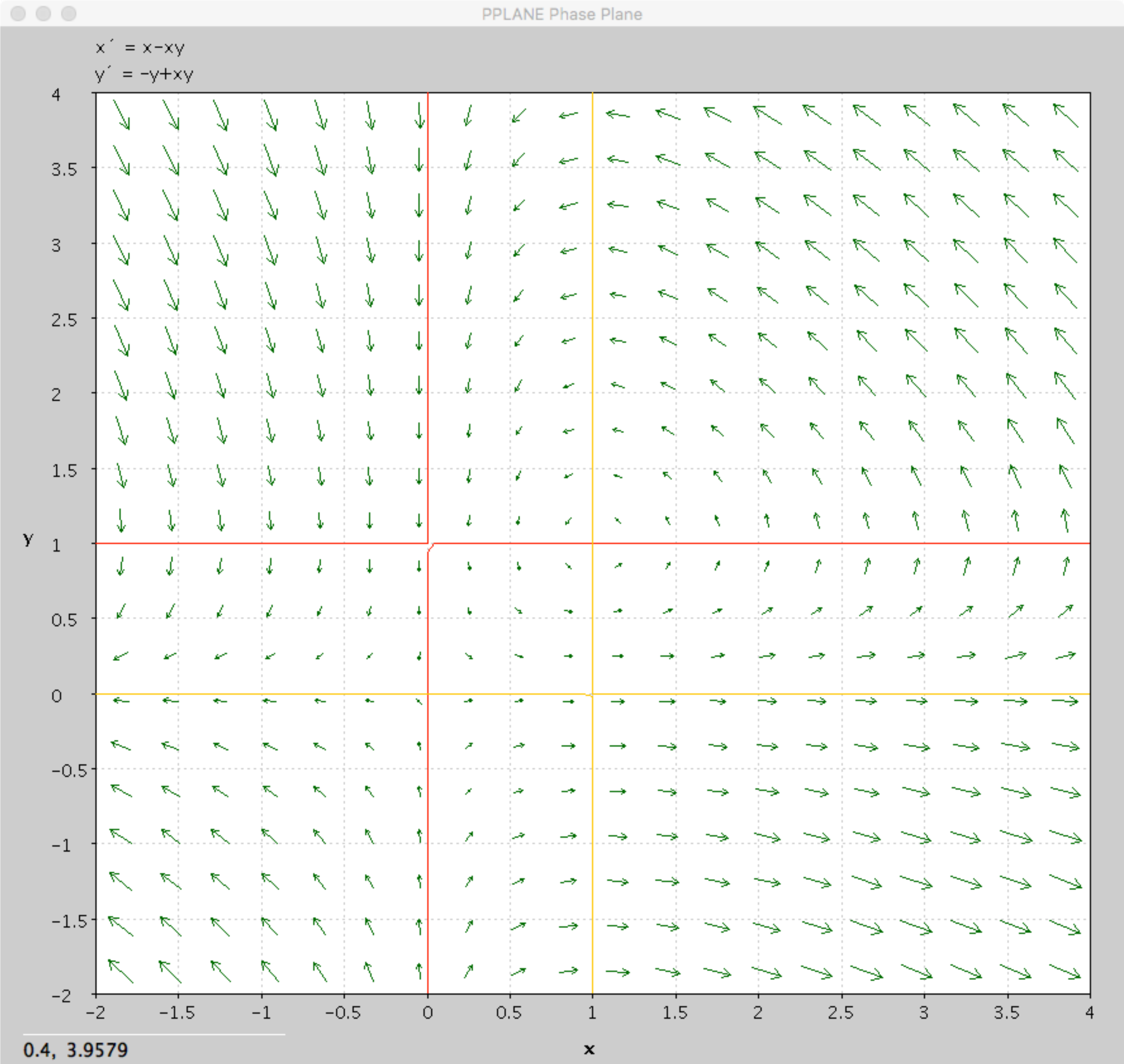
$(x_{ss}, y_{ss}) = (1,1)$

ⓐ rates of change in the region where $x<0$ and $y>1$ ~~is~~ are

$\dot{x} = x(1-y) > 0$ because $x<0$ and $1-y<0$

$\dot{y} = y(x-1) < 0$ " $y>0$ and $x-1<0$

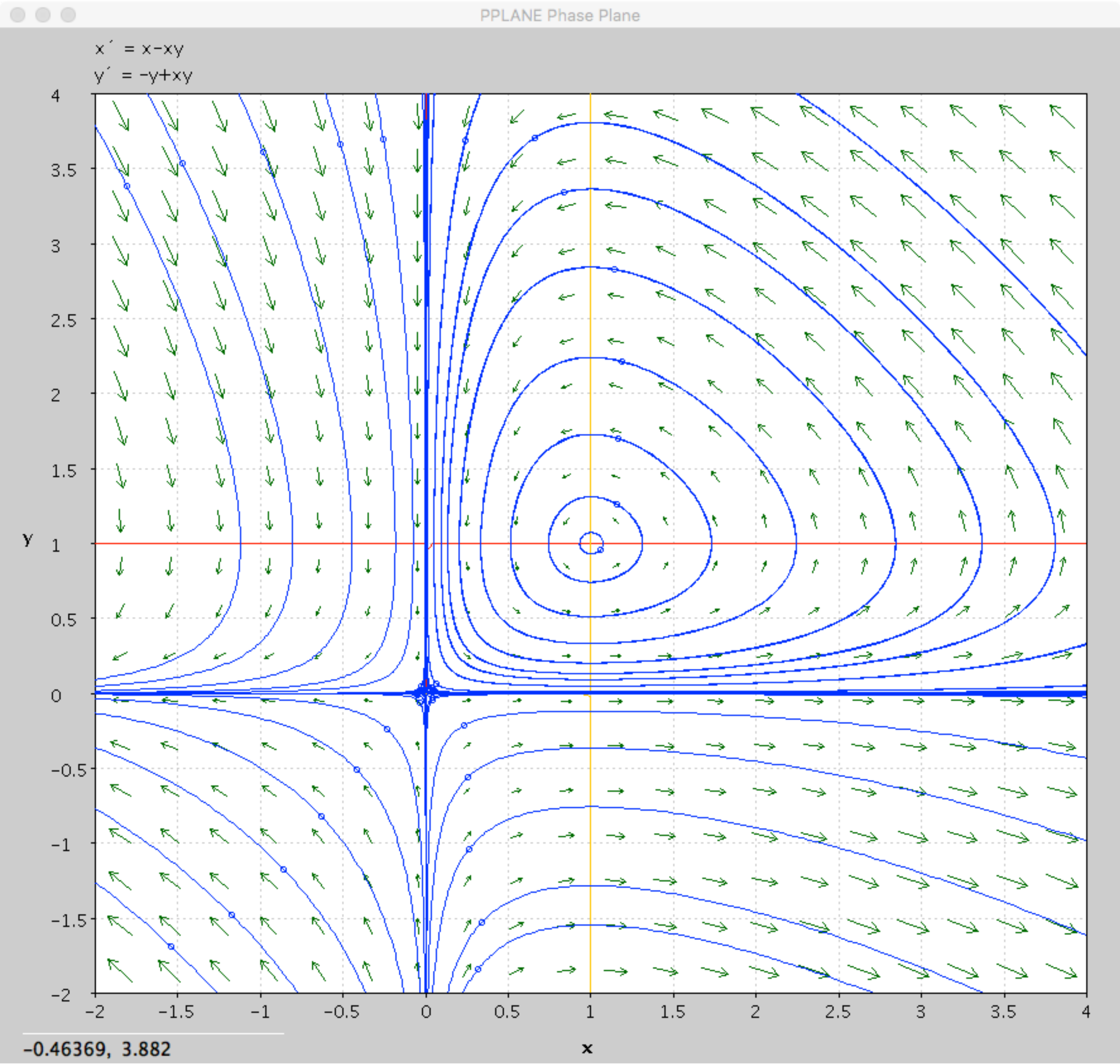
similar arguments give ↑→ throughout entire plane



$x' = x - xy$

$y' = -y + xy$

0.4, 3.9579



PPLANE Phase Plane

x´ = x−xy

y´ = −y+xy

−0.46369, 3.882

linear stability analysis

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 1-y & -x \\ y & x-1 \end{pmatrix}$$

Jacobian

- For the steady-state at $(x_{ss}, y_{ss}) = (0,0)$, J evaluates to

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{aligned} \operatorname{trace} J &= 1-1=0 \\ \det J &= 1(-1) - 0\cdot 0 = -1 \end{aligned}$$

det J is negative so this is <u>unstable</u>

it is in fact a saddle point.

- For the steady-state at $(x_{ss}, y_{ss}) = (1,1)$,

$$J(1,1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \begin{aligned} \operatorname{trace} J &= 0+0 = 0 \\ \det J &= 0\cdot 0 - (-1)1 = 1 \end{aligned}$$

this is also <u>unstable</u>.

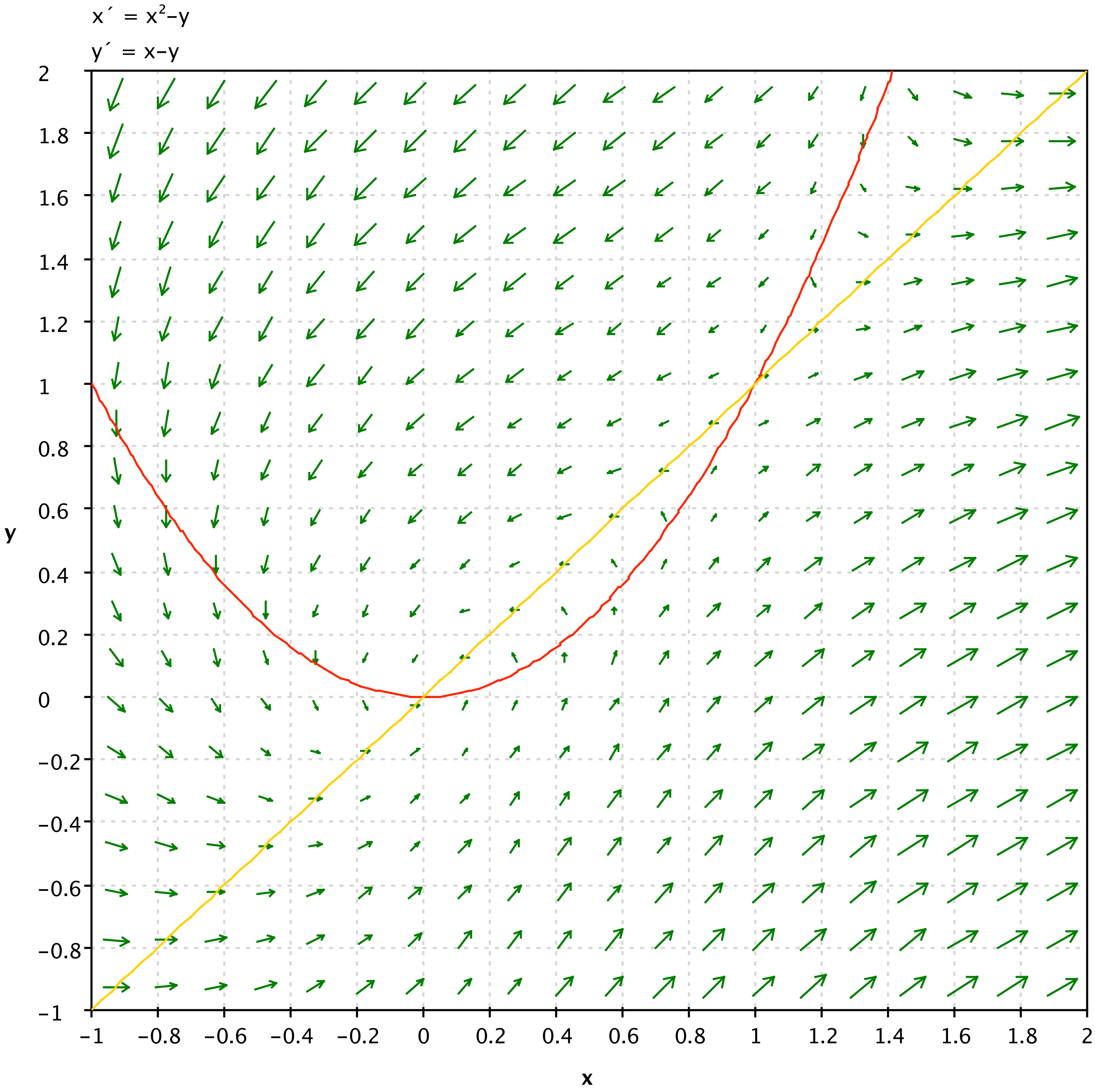
it is a <u>center</u>.

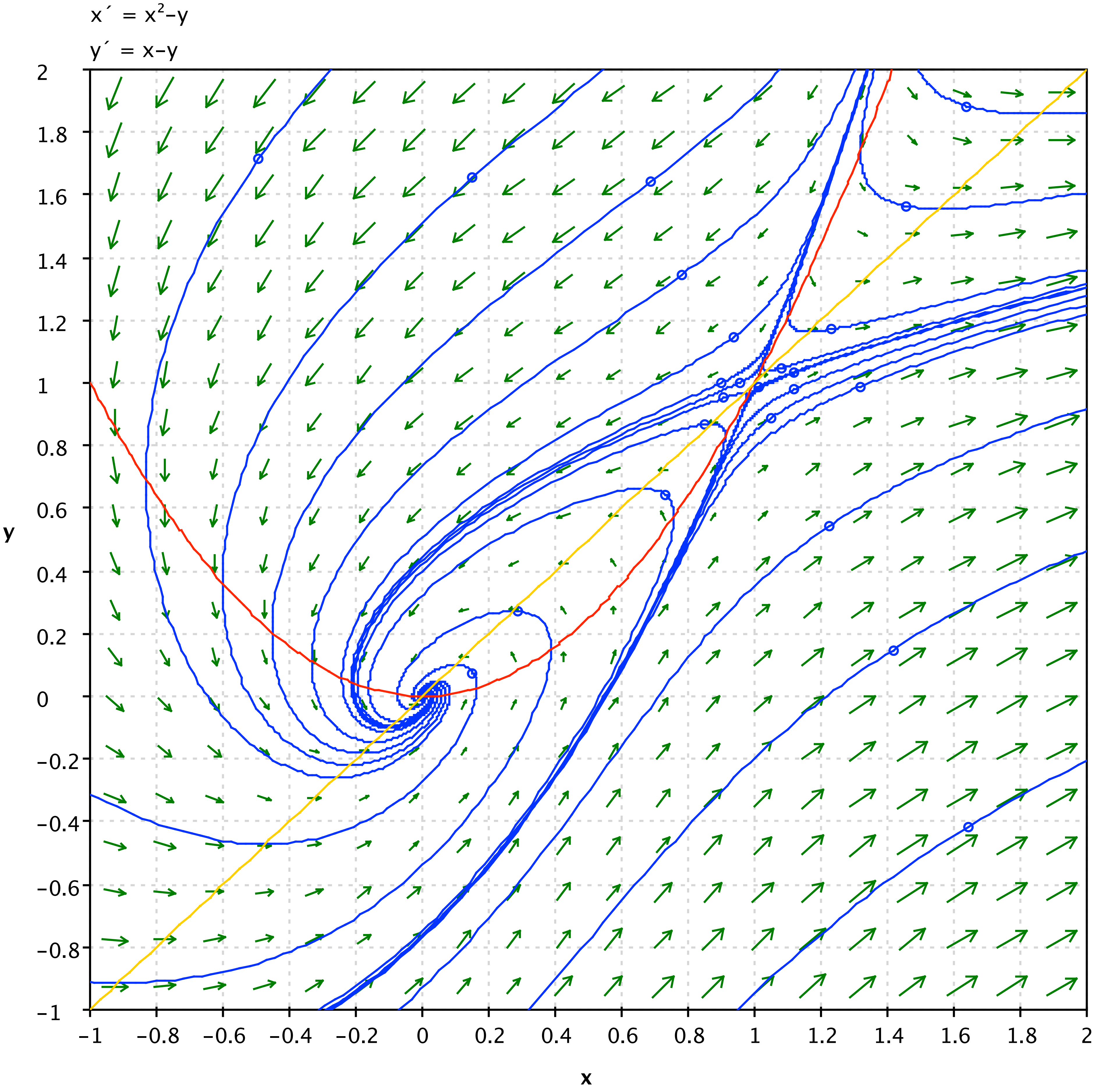
Another example of phase plane analysis

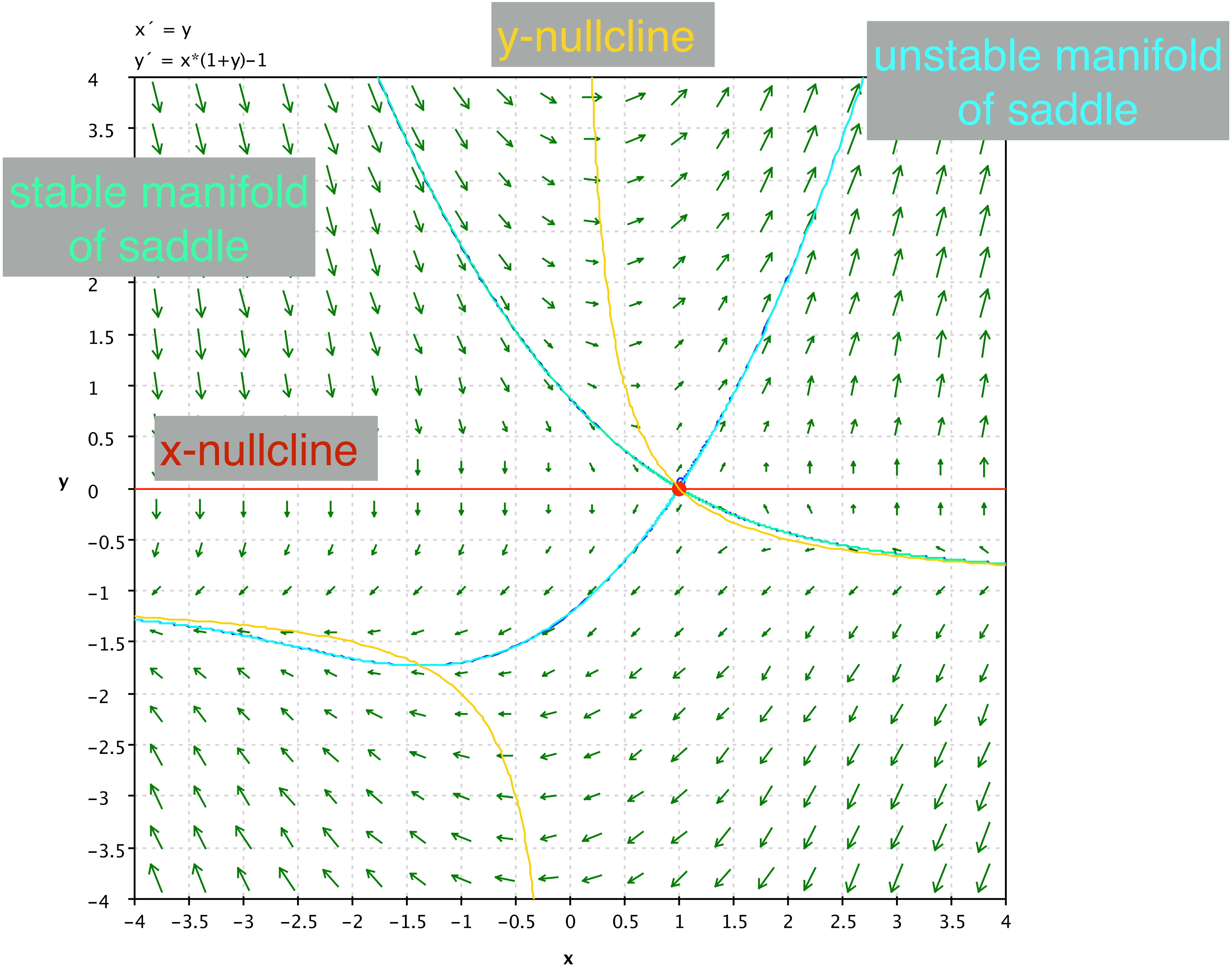
$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = x(1 + y) - 1$$

Can you show (analytically) that
there is one unstable steady state (saddle)


$x' = x^2 - y$
$y' = x - y$
y
x
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


x´ = x²−y
y´ = x−y
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
y
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x

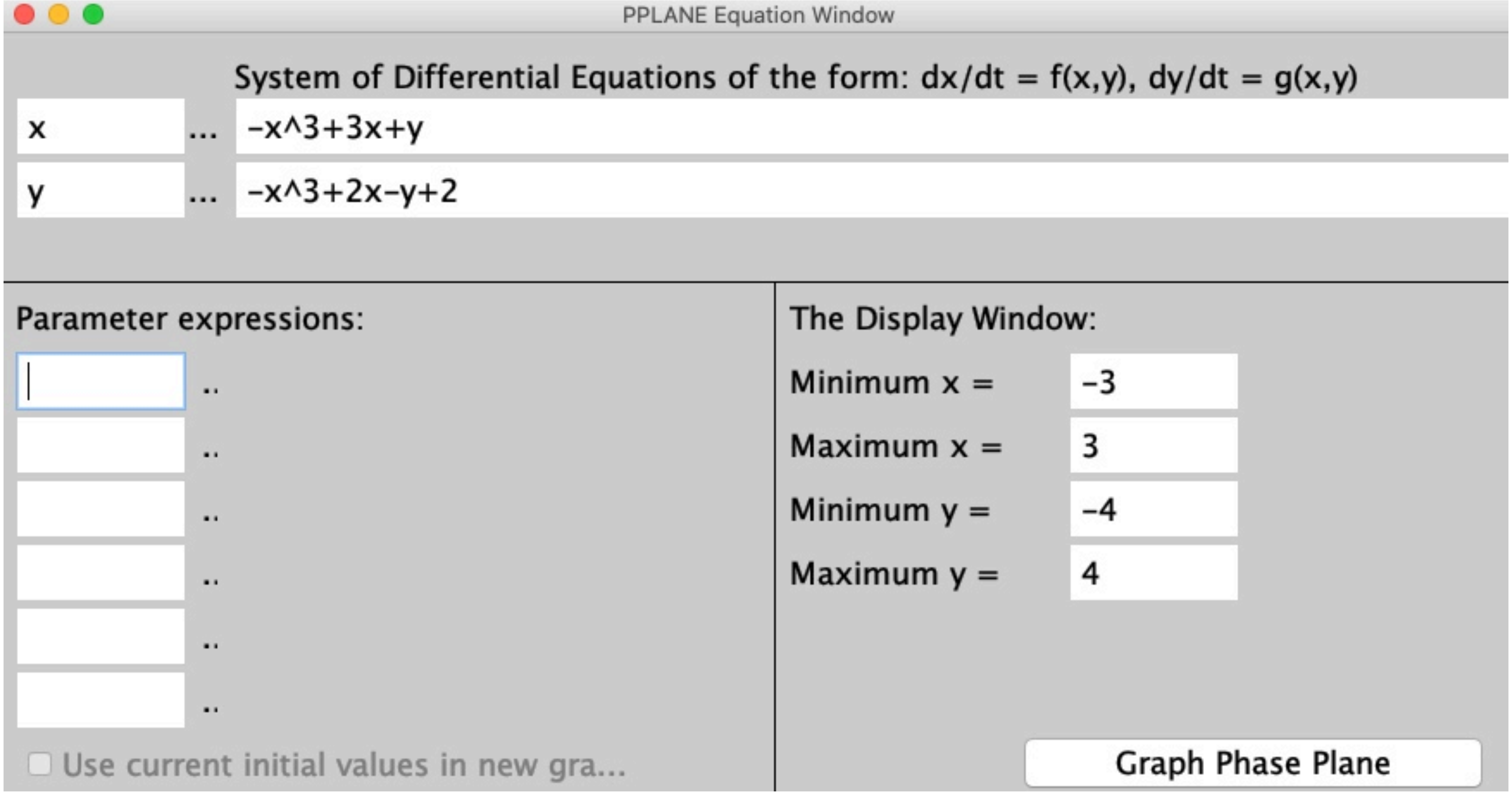


y-nullcline

unstable manifold of saddle

stable manifold of saddle

x-nullcline



System of Differential Equations of the form: dx/dt = f(x,y), dy/dt = g(x,y)

x ... -x^3+3x+y

y ... -x^3+2x-y+2

Parameter expressions:

☐ Use current initial values in new gra...

The Display Window:

Minimum x = -3

Maximum x = 3

Minimum y = -4

Maximum y = 4